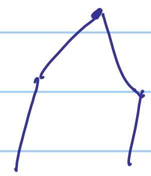
Lecture 12

Note Title 2018-01-30

Branching processes w/ generating fcns.

Recall



$Z_0 = 1$

$Z_1 = 2$

$Z_2 = 2$

Z_n = size at generation n

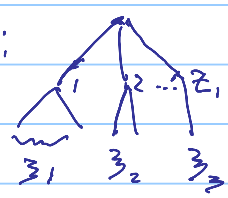
ξ = r.v. $\stackrel{dist}{=}$ # of children of each individual

Let $G_n(s) = G_{Z_n}(s) = \sum_{j=0}^{\infty} P(Z_n = j)\cdot s^j$ ← complicated

Note: $G_1(s) = G_\xi(s)$

Q: $G_2(s) = ?$

A:



$$Z_2 = \underbrace{\xi_1 + \xi_2 + \dots + \xi_{Z_1}}_{\text{iid w/ dist } \xi}$$

Z_1 ← r.v.

$$G_{Z_2}(s) = G(\xi_1 + \xi_2 + \dots + \xi_{Z_1}) = G_{Z_1}(G_\xi(s))$$

$$= G_\xi(G_\xi(s)) = G_1 \circ G_1(s)$$

Q: $G_n(s) = ?$

Thm $$\boxed{G_n(s) = \underbrace{G_1 \circ G_1 \circ \dots \circ G_1(s)}_{\substack{n \text{ compositions} \\ \text{of simple fcns!}}}}$$

Corr $$\boxed{G_{n+m}(s) = G_n \circ G_m(s)}$$

From here on, set $G(s) = G_1(s)$.

This simple characterization of gen. fcns allows us to easily understand the behavior of a branching process:

$$\boxed{\begin{array}{l} \textbf{Prop (Mean \& variance of } Z_n\textbf{)} \\ \text{Let } \mu := \mathbb{E}\xi, \ \sigma^2 = \mathrm{Var}(\xi). \quad \text{Then} \\ (1)\ \mathbb{E} Z_n = \mu^n \\ (2)\ \mathrm{Var}(Z_n) = \begin{cases} n\sigma^2 & \text{if } \mu = 1 \\ \dfrac{\sigma^2(\mu^n - 1)\mu^{n-1}}{\mu - 1} & \text{if } \mu \neq 1 \end{cases} \end{array}}$$

Pf of (1): Recall $\mathbb{E} Z_n = G_n'(1)$.

We have $G_n(s) = G(G_{n-1}(s))$

$$\Rightarrow G_n'(s) = G'(G_{n-1}(s)) \cdot G_{n-1}'(s),$$

$$\Rightarrow G_n'(1) = G'(\underbrace{G_{n-1}(1)}_{1})\, G_{n-1}'(1)$$

$$= G'(1) \cdot G_{n-1}'(1) = \mu \cdot \mathbb{E} Z_{n-1}$$

Iterate and use $\mathbb{E} Z_0 = Z_0 = 1$, to get $\mathbb{E} Z_n = \mu^n$

Pf of (2): Recall: $G_n''(1) = \mathrm{Var}(Z_n) - \mathbb{E} Z_n + (\mathbb{E} Z_n)^2$

$$= \mathrm{Var}(Z_n) - \mu^n + \mu^{2n}$$

Differentiate $G_n'(s)$:

$$G''(s) = \frac{d}{ds} G'(s) = \frac{d}{ds} G'(G_{n-1}(s)) \cdot G_{n-1}'(s)$$

$$= G''(G_{n-1}(s)) \cdot G_{n-1}'(s) \cdot G_{n-1}'(s) + G'(G_{n-1}(s)) \cdot G_{n-1}''(s)$$

$$\Rightarrow G_n''(1) = G''(1) \cdot (G_{n-1}'(1))^2 + G'(1) \cdot G_{n-1}''(1)$$

$$= (\sigma^2 - \mu + \mu^2) \cdot (\mu^{n-1})^2 + \mu \cdot G''_{n-1}(1)$$

Case 1 ($\mu = 1$) Then $G''_n(1) = Var(Z_n) \cancel{-1+1}$

$\Rightarrow G''_n(1) = \sigma^2 + G''_{n-1}(1)$

By iteration $G''_n(1) = \sigma^2(n-1) + \underbrace{\cancel{G''_1(1)}}_{\cancel{Var(\xi)}\ \sigma^2} = \sigma^2 n$

Case 2 ($\mu \neq 1$) Exercize for you.

We can also determine survival probability.

Let $\rho = P(\text{ultimate extinction})$ so survival prob $= 1 - \rho$.

Now, $G_n(s) = 1 \cdot P(Z_n = 0) + s \cdot P(Z_n = 1) + s^2 \cdot P(Z_n = 2) + \ldots$

$\Rightarrow G_n(0) = P(Z_n = 0)$

$$P(\text{ultimate extinction}) = \{Z_n = 0 \text{ for } n \text{ high enough}\}$$
$$= \bigcup_{n=0}^{\infty} \{Z_n = 0\}$$

increasing sets i.e. $\{Z_n = 0\} \Rightarrow \{Z_{n+1} = 0\}$

Thus, $P(\text{ultimate extinction}) = \lim_{n\to\infty} P(Z_n = 0) = \lim_{n\to\infty} G_n(0)$.

Thm $\rho = P(\text{ultimate extinction}) = \left(\begin{array}{l}\text{smallest non-negative} \\ \text{root of } s = G(s)\end{array}\right)$.

Also, let $\mu = \mathbb{E}\xi$, $\sigma^2 = Var(\xi)$.

Then $\rho = 1$ if $\mu < 1$ & $\rho < 1$ if $\mu > 1$.

If $\sigma^2 > 0$ & $\mu = 1$, then $\rho = 1$.