Lecture 11

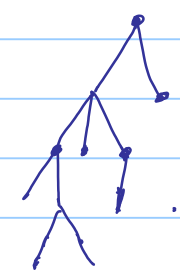
Note Title 2018-01-28

Branching processes.

- Population evolves in generations,
- individual has random # of offspring, indep. of other individuals and with the same distribution
  - Offspring is a r.v. ξ. Let $p_j = P(\xi = j)$ $j = 0, 1, \dots$ and suppose $p_0 > 0$

  non-zero chance to die w/ no offspring.

Let Z_n = # of individuals in generation n.
Assume $Z_0 = 1$.



$Z_0 = 1$
$Z_1 = 2$
$Z_3 = 3$
$Z_4 = 3$
$Z_5 = 2$

Q: • Does the population eventually die out w/ prob 1?
• What is the prob of survival?

$(Z_n)_{n \geq 0}$ is a M.C.

- State space $\{0, 1, \dots\}$
- State 0 is recurrent (absorbing)
- For $i > 0$, stat i is transient since $P_{i0} = p_0^i > 0$.

  (non zero chance to jump to state 0. Will never return.)

- Since any finite set of transient states can only be visited a finite # of times, either Z_n is eventually 0, or $Z_n \to \infty$.

Transition probabilities are a bit messy. To study survival prob, we use a simplifying transformation:

Def For a r.v. $\xi \in \{0,1,\dots\}$, the generating function $G_\xi = G : \mathbb{R} \to \mathbb{R}$ satisfies

$$G(s) = \mathbb{E}\, s^\xi = \sum_{j=0}^{\infty} s^j P(\xi = j)$$

sometimes omitted

Ex) $\xi = \begin{cases} 0 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases}$ $\qquad G(s) = \frac{1}{2}s^0 + \frac{1}{2}s^1 = \frac{1}{2} + \frac{1}{2}s.$

Properties of generating functions:

1) If X, Y are indep r.v.'s, then
$G_{X+Y}(s) = G_X(s) \cdot G_Y(s)$

Pf. $G_{X+Y}(s) = \mathbb{E}\, s^{X+Y} = \mathbb{E}\, s^X \cdot s^Y \underset{\text{indep}}{=} \mathbb{E}\, s^X \cdot \mathbb{E}\, s^Y = G_X(s) \cdot G_Y(s)$ ✓

2) $X_1, X_2, \dots$: iid r.v.'s w/ gen. fcn G_X.
N: indep of $X_1, X_2, \dots$ & w/ gen. fcn G_N.
$T = X_1 + \dots + X_N$ (random # of summands)
$\Rightarrow G_T(s) = G_N(G_X(s))$

Pf: $G_T(s) = \mathbb{E}\, s^T = \mathbb{E}\left[\mathbb{E}\, s^T / N\right] = \sum_{n=0}^{\infty} \mathbb{E}\left[s^T / N = n\right] \cdot P(N = n)$

$= \sum_{n=0}^{\infty} \mathbb{E}\left[s^{X_1 + \dots + X_N}\right] \cdot P(N = n) = \sum_{n=0}^{\infty} \left(G_X(s)\right)^n \cdot P(N = n)$

$= G_N(G_X(s)).$

3) Since $G_X(s) = \sum_{j=0}^{\infty} s^j P(X = j)$

a) $G_X(1) = \sum_{j=0}^{\infty} P(X = j) = 1$

b) $G'_X(s) = \sum_{j=0}^{\infty} j\, s^{j-1} \cdot P(X = j) \Rightarrow G'_X(1) = \sum_{j=0}^{\infty} j P(X = j) = \mathbb{E}\, X$

c) $G_X''(s) = \sum_{j=0}^{\infty} j(j-1)\, s^{j-2}\, P(X=j)$

$\Rightarrow G_X''(1) = \sum_{j=0}^{\infty} (j^2 - j) P(X=j) = \mathbb{E}X^2 - \mathbb{E}X$

$\Rightarrow Var(X) = G_X''(1) + G_X'(1) - (G_X'(1))^2.$