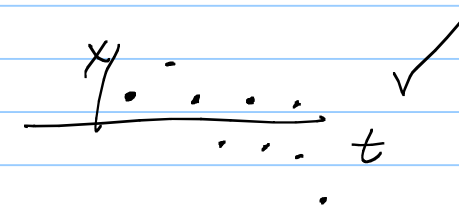
Note Title 2018-01-02

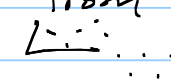
Lecture 1 Math 303: Introduction to stochastic proc.

- Course outline web page: www.yanivplan.com/303-section-201
- Warm up example

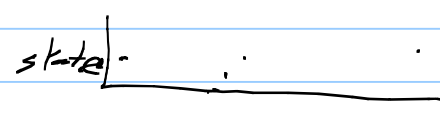
We will study discrete stochastic processes:



Systems with a finite, or countable, number of states. Move randomly from one state to another at - fixed times ≈ ½ the class



or - continuously random times



Key assumptions:
- Markov property: Prob dist for next state depends on current state but not previous states
- Stationarity: time homogeneity of jump probabilities.

Applications:
- physics (diffusion of particles in space)
- biology (population evolving under random births/deaths...)
- economics (stock market)
- gambling
- data science (page Rank)

Also, the math is interesting (Calculus, lin. algebra, probability) and we will gain a clear understanding of our central objects of study: Markov chains and Markov processes.

<u>Markov chains:</u>

- state space is $\{0,1,\dots,N\}$ or $\{0,1,2,\dots\}$
- X_n is the state of the system at time $n=0,1,2,\dots$

Main assumptions:

$$\left.\begin{aligned} &P(X_{n+1}=j \mid X_n=i_n, X_{n-1}=i_{n-1}, \dots X_0=i_0) \\ &= P(X_{n+1}=j \mid X_n=i_n) && [\text{Markov property}] \\ &= P_{ij} && [\text{Independent of } n \text{ by stationarity}] \end{aligned}\right\} \forall n, i_0, i_1, \dots, i_n$$

Ex) 2-state Markov Chain: $X_n = (\text{state on day } n) \in \{\overset{\text{broken}}{0}, \overset{\text{working}}{1}\}$

Machine broken or working.

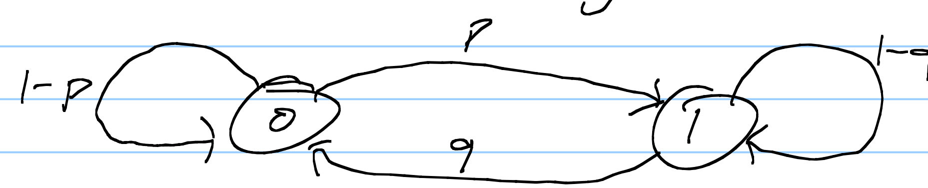
Let $p = P_{01} = P(X_{n+1} = 1 \mid X_n = 0)$ = prob fixed tomorrow if broken today

$q = P_{10} = P(X_{n+1} = 0 \mid X_n = 1)$ = " broken " " working "

Note: must have $P_{01} + P_{00} = 1$, $P_{10} + P_{11} = 1$, so in matrix form:

transition matrix $\rightarrow P = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$

Transition diagram



Initial state: Let $\alpha_i = P(X_0 = i)$ $i = 0, 1$ $\rightarrow$ $\alpha_0 + \alpha_1 = 1$.

Q: What is the steady state dist$\underline{n}$, i.e., $\pi_i = \lim_{n\to\infty} P(X_n = i)$? (if lim exists)

<u>Recursion trick:</u>

$$P(X_{n+1}=0) = P(X_{n+1}=0 \mid X_n=0)\cdot P(X_n=0) + P(X_{n+1}=0 \mid X_n=1)\cdot P(X_n=1)$$

$$= (1-p)\cdot P(X_n=0) + q(1-P(X_n=0)) \qquad (*)$$

Assuming lim exists, take $\lim_{n\to\infty}$ of both sides of $(*)$ to give

$$\pi_0 = (1-p)\pi_0 + q(1-\pi_0) \Rightarrow \pi_0 = \frac{q}{p+q} \Rightarrow \pi_1 = \frac{p}{p+q}$$

<u>Brute force solution</u> (without assuming lim exists):

Let $A_n = P(X_n=0)$ so $(*)$ becomes $A_{n+1} = (1-p-q)A_n + q = aA_n + q$

$n=0$: $\quad A_1 = a \cdot A_0 + q = a\alpha_0 + q$

$n=1$ $\quad A_2 = a \cdot A_1 + q = a^2\alpha_0 + aq + q$

$\vdots$

$n=3$ $\quad A_3 = aA_2 + q = a^3\alpha_0 + a^2 q + aq + q$

$\vdots$

n $\quad A_n = a^n \alpha_1 + q \sum_{j=0}^{n-1} a^j$

$$= a^n \alpha_1 + q \frac{1-a^n}{1-a} \qquad (\text{assuming } a \neq 1 \text{ i.e. } p+q \neq 0.)$$

$$= a^n \left(\alpha_1 - \frac{q}{p+q}\right) + \frac{q}{p+q}$$

Suppose $p+q \neq 2$ so $|a|<1$. $\Rightarrow a^n \to 0$ & thus

$$\pi_0 = \lim_{n\to\infty} A_n = \frac{q}{p+q}$$

$$\pi_1 = 1 - \pi_0 = \frac{p}{p+q}$$

<u>Rmk</u>: π_0, π_1 are indep of α_0, α_1!

Also, one can show that if we start in the limiting state i.e, $\alpha_0 = \pi_0$, $\alpha_1 = \pi_1$, then

$$P(X_n = 0) = P(X_{n-1} = 0) = \dots = \alpha_0$$

$$P(X_n = 1) = P(X_{n-1} = 1) = \dots = \alpha_1$$

This dist$^{\underline{n}}$ (π_0, π_1) is called the <u>invariant</u>, or <u>equilibrium</u>, or <u>steady state</u> or <u>stationary</u> dist$^{\underline{n}}$.

Note: If $(\alpha_0, \alpha_1) \neq (\pi_0, \pi_1)$ the system goes to eq$\underline{m}$ expon. fast

(error decays prop. to $|a|^n$).