Lecture 10

Note Title 2018-01-25

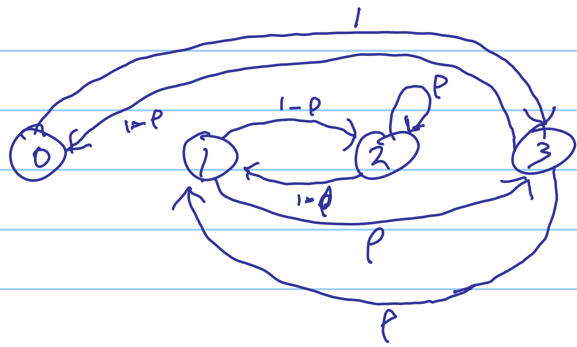
More examples of reversible MC's.

Ex) (chap 4, #46) Smith has 3 umbrellas (total) at home & at casino.

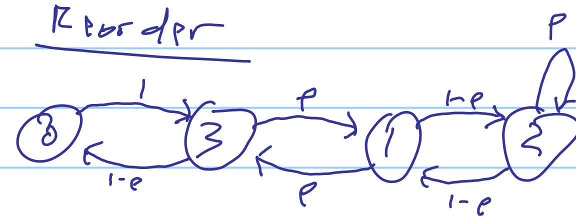
- takes an umbrella if raining
- doesn't take one if not raining (or if $\nexists$ umbrella)
- $\mathbb{P}(\text{rain}) = \rho$

Q: What fraction of time does Smith get wet?

X_n: # of umbrellas at Smith's location.



Reorder



Irreducible, finite state, RW $\to$ Reversible $\Rightarrow$ unique stat dist π.

Proportion of wet trips: $\pi_0 \cdot \rho$ (no umbrella) (rainy)

Find π: Solve $\pi_i \cdot P_{ij} = \pi_j \cdot P_{ji}$ (since it is reversible!)

$$\left.\begin{aligned} \pi_0 \cdot \cancelto{1}{P_{03}} &= \pi_3 \cdot \cancelto{1-\rho}{P_{30}} \\ \pi_3 \cdot \cancelto{\rho}{P_{31}} &= \pi_1 \cdot \cancelto{\rho}{P_{13}} \\ \pi_1 \cdot \cancelto{1-\rho}{P_{12}} &= \pi_2 \cdot \cancelto{1-\rho}{P_{21}} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \pi_0 &= (1-\rho)\pi_3 \\ \pi_3 &= \pi_1 \\ \pi_1 &= \pi_2 \end{aligned}\right\} \Rightarrow \begin{aligned} \pi_1 &= \pi_2 = \pi_3 \\ \pi_0 &= (1-\rho)\pi_1 \end{aligned}$$

Also, $\sum \pi_i = 1 \Rightarrow 3 \cdot \pi_1 + (1-\rho)\pi_1 = 1$

$\Rightarrow \quad \pi_1 = \dfrac{1}{4-p} = \pi_2 = \pi_3, \qquad \pi_0 = \dfrac{1-p}{4-p}$

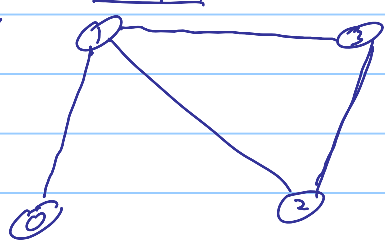
$\Rightarrow$ Frac of wet trips $= p \cdot \pi_0 = \dfrac{p(1-p)}{4-p} =: f(p)$

Eg) $f\left(\frac{1}{2}\right) = \frac{1}{14}$

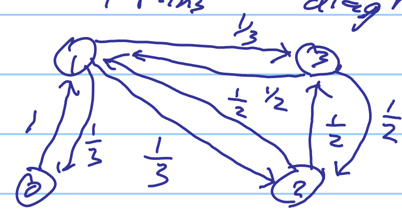
Worst case p? Set $f'(p) = 0$ to give $p = .5359$ and $f(p) = .072$.

Ex) MC on graph: Let d_i be the # of edges at vertex i. Travel to each neighbor with prob $\frac{1}{d_i}$.

Graph



trans diagram



Reversible? Maybe. Let's guess that it is and check if $\pi_i P_{ij} \stackrel{?}{=} \pi_j P_{ji}$, $\sum \pi_i = 1$ has a soln.

$$P_{ij} = \begin{cases} \frac{1}{d_i} & \text{if } i,j \text{ share edge} \\ 0 & \text{else} \end{cases}$$

Thus, $(*)$ becomes

$$\begin{cases} \pi_i \cdot 0 = \pi_j \cdot 0 & \text{if no shared edge} \\ \pi_i \cdot \frac{1}{d_i} = \pi_j \cdot \frac{1}{d_j} & \text{if shared edge} \end{cases}$$

$\Rightarrow$ Take $\pi_i = \lambda d_i$ for some $\lambda > 0$.

We also need $\sum_i \pi_i = 1$ i.e. $\sum_i \lambda d_i = 1$

$$\Rightarrow \lambda = \frac{1}{\sum_i d_i}$$

Thus, it is reversible w/ stat. dist.

$$\boxed{\pi_i = \frac{d_i}{\sum_i d_i}}$$