Slepian's $\neq$. Sharp bounds $\sigma_1(A)$, $\sigma_n(A)$.

Note Title 2015-10-09

Important assumption today: A has iid $N(0,1)$ entries.

Background: Some useful facts about (centered) Gaussian vectors and processes.

- <u>Multivariate Gaussian vector:</u>

$g=(g_1, g_2, \dots, g_n) \sim N(0, \Sigma)$: $\Sigma = \mathbb{E}gg^T$ is the covariance matrix.

Density: $\frac{1}{(\sqrt{2\pi})^n |\Sigma|} \exp(-\frac{1}{2} x^T \Sigma^{-1} x)$

- <u>Linear transformation:</u>

Let $g \sim N(0, \Sigma)$. Let $A \in \mathbb{R}^{m \times n}$ be a <u>fixed</u> matrix.

Then $Ag \sim N(0, A\Sigma A^T)$

In particular, if $AA^T = I_m$, and $g \sim N(0, I_n)$, then $Ag \sim N(0, I_m)$. (rotation invariance.)

- <u>Whitening:</u> If $g \sim N(0, \Sigma)$ then $\Sigma^{-\frac{1}{2}} g \sim N(0, I_n)$

- <u>Linear funtional:</u> If $g \sim N(0, \Sigma)$, $x \in \mathbb{R}^n$, then $\langle g, x \rangle \sim N(0, x^T \Sigma x)$

(Alternative definition of multivariate Gaussian: For any $x \in \mathbb{R}^n$, $\langle g, x \rangle$ is Gaussian)

(Note: If g_1, g_2 are both Gaussian $g = (g_1, g_2)$ is not necessarily multivariate Gaussian. Find an example!)

- <u>Independence = no correlation</u>

If g is multivariate Gaussian, then g_i is indep. of g_j iff $\mathbb{E} g_i g_j = 0$.

Gaussian process: We call X_t, $t \in T$, a Gaussian process if for any finite subset $\{t_1, t_2, \dots, t_n\} \subseteq T$, $(X_{t_1}, X_{t_2}, \dots X_{t_n})$ is multivariate Gaussian.

Observe: A Gaussian process is a kind of sub-Gaussian process.

Ex) Let $g \sim N(0, I_n)$. Bound $\|g\|_2$ the hard way! (Using Dudley's $\neq$)

$$\|g\|_2 = \sup_{v \in S^{n-1}} \langle g, v \rangle = \sup_{t \in S^{n-1}} \underbrace{X_t}_{\text{Gaussian process}}$$

Sub-Gaussian increments?

$$X_t - X_s = \langle g, t \rangle - \langle g, s \rangle = \langle g, t-s \rangle \sim N(0, \|t-s\|_2^2)$$

$$\sim \|t-s\|_2 \, N(0,1)$$

$$\Rightarrow \|X_t - X_s\|_{\psi_2} = \|t-s\|_2 \cdot \|N(0,1)\|_{\psi_2}$$

$$= c\|t-s\|_2 \leftarrow \text{associated metric}$$

Thus, by Dudley's $\neq$,

$$\mathbb{E}\|g\| \le c \int_0^2 \sqrt{\log N(S^{n-1}, \|\cdot\|_2, \epsilon)} \, d\epsilon$$

$$\le c \int_0^2 \sqrt{\log \left(\frac{c}{\epsilon}\right)^n} \, d\epsilon = c\sqrt{n}$$

(Optimal bound up to abs const.)

1.3 Slepian $\neq$, Gordon $\neq$, sharp bound for extreme singular values of Gaussian A.

Let $A \in \mathbb{R}^{m \times n}$ have $N(0,1)$ entries. $m \geq n$

Thm (Gordon '85)

$$\boxed{\sqrt{m}-\sqrt{n} \le \mathbb{E}\,\sigma_n(A) \le \mathbb{E}\,\sigma_1(A) \le \sqrt{m}+\sqrt{n}}$$

We bound $\sigma_1(A)$ w/ the following comparison $\neq$:

Lemma (Slepian's inequality)

Consider two centered Gaussian processes $(X_t)_{t\in T}$, and $(Y_t)_{t\in T}$. Assume that their increments satisfy

$$\mathbb{E}(X_s-X_t)^2 \le \mathbb{E}(Y_s-Y_t)^2 \qquad \forall\ s,t\in T.$$

Then

$$\mathbb{E}\sup_{t\in T} X_t \le \mathbb{E}\sup_{t\in T} Y_t$$

(Useful when $\mathbb{E}\sup Y_t$ is easier to bound.)

Proof (that $\mathbb{E}\,\sigma_1(A) \le \sqrt{m}+\sqrt{n}$)

$$\sigma_1(A) = \sup_{u,v\in S^{n-1}} u^T A v =: \sup_{u,v\in S^{n-1}} X_{u,v}$$

- Compute increments: (Note: $X_{uv} = u^T A v = \langle A, uv^T\rangle$)

$$\mathbb{E}(X_{u,v}-X_{w,z})^2 = \mathbb{E}\left(\langle A,uv^T\rangle - \langle A,wz^T\rangle\right)^2$$

$$= \mathbb{E}\langle A, uv^T - wz^T\rangle^2$$

$$= \|uv^T - wz^T\|_F^2$$

$$\le \|u-w\|_2^2 + \|v-z\|_2$$

$\boxed{\text{HW}}$ Prove this $\neq$

Now, find a simpler Gaussian process

that has these increments:

$$Y_{u,v} = \langle g, u\rangle + \langle h, v\rangle \qquad \text{where } g \sim N(0, I_m), \; h \sim N(0, I_n)$$

Thus, by Slepian's ≠,

$$\mathbb{E}\,\sigma_1(A) = \mathbb{E} \sup_{u,v \in S^{n-1}} X_{u,v} \leq \mathbb{E} \sup_{u,v \in S^{n-1}} Y_{u,v}$$

$$= \mathbb{E} \sup_{u \in S^{n-1}} \langle g, u\rangle + \mathbb{E} \sup_{v \in S^{n-1}} \langle h, v\rangle$$

$$= \mathbb{E}\|g\|_2 + \mathbb{E}\|h\|_2$$

$$\leq \sqrt{\mathbb{E}\|g\|_2^2} + \sqrt{\mathbb{E}\|h\|_2^2} \qquad \text{(By Jensen's ≠)}$$

$$= \sqrt{m} + \sqrt{n}$$

QED

(Similar argument to control $\sigma_n(A) = \inf_{u \in S^{n-1}} \sup_{v \in S^{n-1}} u^T A v$.
We would replace Slepian ≠
w Gordon ≠. See [Vershynin. Introduction to RMT])