Math 303 Assignment 4: Due Friday, February 9 at start of class

I. Problems to be handed in:

1. Is it possible for a branching process to be reversible? If so, what must ξ satisfy? (Recall, each individual has a number of children distributed as a random independent copy of ξ.)

2. Let $(X_0, X_1, \ldots,)$ be a reversible Markov chain with state space S, transition matrix P, and stationary distribution π. Show that if the chain is started with initial distribution π, then for any n and any $s_{i_0}, s_{i_1}, \ldots, s_{i_n} \in S$, we have

$$P(X_0 = s_{i_0}, X_1 = s_{i_1}, \ldots, X_n = s_{i_n}) = P(X_n = s_{i_0}, X_{n-1} = s_{i_1}, \ldots, X_0 = s_{i_n}).$$

   In other words, the chain is equally likely to make a tour through the states $s_{i_0}, s_{i_1}, \ldots, s_{i_n}$ in forwards or backwards order.

3. Determine the generating function $G_X(s) = Es^X$, for each of the following random variables.

   (a) $X \sim \textit{Bernoulli}(p)$, i.e., $P(X = 1) = p$, $P(X = 0) = 1 - p$.

   (b) $X \sim \textit{Binomial}(n, p)$.

   (c) $X \sim \textit{Poisson}(\mu)$, i.e., $P(X = j) = e^{-\mu}\mu^j/j!$ for $j = 0, 1, 2, \ldots$.

   (d) $X \sim \textit{Geometric}(p)$, i.e, $P(X = j) = (1-p)^{j-1}p$ for $j = 1, 2, \ldots$.

   (e) $X = A + B + C$ where $A \sim \textit{Bernoulli}(p), B \sim \textit{Binomial}(n, p), C \sim \textit{Poisson}(\mu)$.

4. Let $X_1, X_2, \ldots$ be $\textit{Geometric}(p)$ random variables, let $Y_1, Y_2, \ldots$ be $\textit{Poisson}(\mu)$ random variables, and let $M \sim \textit{Binomial}(n, p)$. Assume all random variables above are independent of each other.

   Now let $N = Y_1 + Y_2 + \ldots + Y_M$. Let $X = X_1 + X_2 + \ldots + X_N$. Find the generating function of X.

5. Consider a branching process in which the children of an individual are a copy of the random variable ξ, where ξ satisfies:

$$P(\xi = 0) = \alpha, \quad P(\xi = 1) = 0, \quad P(\xi = 2) = \beta, \quad P(\xi = 3) = 1 - \alpha - \beta.$$

   (a) If $\alpha = \beta = 1/3$, and the population starts with **10 individuals**, find the probability of eventual extinction.

   (b) Again with $\alpha = \beta = 1/3$, and with a population starting with 10 individuals, find the probability of extinction within 3 generations, i.e, $P(Z_3 = 0)$.

   (c) Under the constraint that $E\xi = 2$, find α and β to maximize the probability of extinction.

II. Recommended problems: These provide additional practice but are not to be handed in.
Textbook Chapter 4: Exercises 64, 66, 71, 73.