Math 303 Assignment 3: Due Wednesday, January 31 at start of class

I. Problems to be handed in:

1. Let C be a communicating class of a Markov chain. We say that C is *closed* if $P_{ij} = 0$ for all states $i \in C$ and $j \notin C$. In other words, a communicating class is closed if there is no escape from that class.

   (a) Show that a finite communicating class C is closed if and only if its states are recurrent.

   (b) Find an example of a Markov chain with no closed communicating class.

2. Textbook Chapter 4 Exercise 33.

3. Textbook Chapter 4 Exercise 36 (a), (b), (c) with $P_{0,0} = 0.3$ and $P_{0,1} = 0.7$.

4. Consider the random walk on the integers $\{0, 1, 2, 3\}$ which takes steps $+1$ (to the right) with probability $\frac{1}{3}$ and -1 (to the left) with probability $\frac{2}{3}$, except at the endpoints where there is reflection; this means that a step from 1 to 0 is always followed by a step from 0 to 1, and a step from 2 to 3 is always followed by a step from 3 to 2.

   (a) Determine the transition matrix for this Markov chain.

   (b) Argue that the chain is time reversible without considering the detailed balance equations.

   (c) What is the stationary distribution?

   (d) Suppose the Markov chain has been running for a long time. What fraction of time has it spent in state 0?

5. A bishop starts at the bottom left of a chess board and performs random moves. At each stage, she picks one of the available legal moves with equal probability, independently of the earlier moves. Let X_n be her position after n moves. What is the mean number of moves before she returns to her starting square?

   Hint: Read and apply Example 4.36.

6. Let X_n be a Markov chain with state space $\{0, 1, 2, \dots\}$ with transition probabilities

   $$p_{i,i+1} = a_i \quad \text{and} \quad p_{i,0} = 1 - a_i$$

   for all state i, where a_i are numbers between 0 and 1. Let $b_0 = 1$ and $b_i = a_0 a_1 \cdots a_i$. Show that the chain is

   (a) recurrent if and only if $\lim_{i\to\infty} b_i = 0$,

   (b) positive recurrent if and only if $\sum_{i=0}^{\infty} b_i < \infty$.

   (c) Find the stationary distribution in the positive recurrent case.

II. Recommended problems: These provide additional practice but are not to be handed in.

Chapter 4: 32, 42, 62, 47, 57, 68b.

7. (a) Prove that periodicity is a class property: If i, j are communicating states of a Markov chain, then their periods satisfy $d_i = d_j$. Hint: Show that d_i divides d_j.

   (b) Let i be a state of a Markov chain. Prove that the following statements are equivalent:
   (1) The period of i is d,
   (2) There is a positive integer N such that for each $n > N$ we have $P_{i,i}^n > 0$ if and only if d divides n.