Math 302, Assignment 2

(1) In a small town, there are three bakeries. Each of the bakeries bakes twelve cakes per day. Bakery 1 has two different types of cake, bakery 2 three different types, and bakery 3 four different types. Every bakery bakes equal amounts of cakes of each type.
You randomly walk into one of the bakeries, and then randomly buy two cakes.
   (a) What is the probability that you will buy two cakes of the same type?
   (b) Suppose you have bought two different types of cake. Given this, what is the probability that you went to bakery 2?

(2) An assembly line produces a large number of products, of which 1% are faulty in average. A quality control test correctly identifies 98% of the faulty products, and 95% of the flawless products. For every product that is identified as faulty, the test is run a second time, independently.
   (a) Suppose that a product was identified as faulty in both tests. What is the probability that it is, indeed, faulty?
   (b) What if the quality control test is only performed once?

(3) Let m be an integer chosen uniformly from $\{1, \ldots, 100\}$. Decide whether the following events are independent:
   (a) $E = \{m \text{ is even}\}$ and $F = \{m \text{ is divisible by } 5\}$
   (b) $E = \{m \text{ is prime}\}$ and $F = \{\text{at least one of the digits of } m \text{ is a } 2\}$
   (c) Can you replace the number 100 by a different number, in such a way that your answer to (a) changes? (E.g., if your answer was "dependent", try to change the number 100 in such a way your answer becomes "independent").

(4) Let X be a discrete random variable with values in $\mathbb{N} = \{1, 2, \ldots\}$. Prove that X is geometric with parameter $p = \mathbb{P}(X = 1)$ if and only if the *memoryless property*

$$\mathbb{P}(X = n + m \,|\, X > n) = \mathbb{P}(X = m)$$

holds.
Hint: To show that the memoryless property implies that X is geometric, you need to prove that the p.m.f. of X has to be $\mathbb{P}(X = k) = p(1 - p)^{k-1}$. For this, use $\mathbb{P}(X = k) = \mathbb{P}(X = k + 1|X > 1)$ repeatedly.

(5) 6. In a card game, 13 cards are given to you out of a deck of 52. This game is being played 50 times. Identify (with names and parameters) the following random variables:

   (a) The number of games in which all cards you receive have the same suit.
   (b) The first time where the number of aces you receive is at least 1.
   (c) The number of games in which you receive exactly three aces.
   (d) The third time in which you received no aces.

(6) **Challenge, not marked.** You have one million \$, but for some reason want to earn an additional \$50. Your strategy is to play roulette at a casino, and always bet \$1 on black, until you own \$1,000,050. What is the

probability that you will be successful?
Hint: The probability of black in a single game is $p = \frac{18}{38}$. Prove a recursion relation for the probability P_n of finishing at \$1,000,050, starting with \$$n$.

(7) Let X take values $\{1, 2, 3, 4, 5\}$, and p.m.f. given by

TABLE 1. The p.m.f. of X

k	1	2	3	4	5
$\mathbb{P}(X = k)$	1/7	1/14	3/14	2/7	2/7

(a) Calculate $\mathbb{P}(X \leq 3)$
(b) Calculate $\mathbb{P}(X < 3)$
(c) Calculate $\mathbb{P}(X < 4.12 | X > 1.6)$
(d) Calculate $\mathbb{E}\, X$
(e) Calculate $\mathbb{E}|X - 2|$

(8) Consider the following lottery: There are a total of 10 tickets, of which 5 are "win" and 5 are "lose". You draw tickets until you draw the first "win". Drawing one ticket costs \$2, 2 tickets \$4, 3 tickets \$8, and so on. A winning ticket pays out \$8.
(a) Let X be the number of tickets you draw in the lottery (i.e. the number of tickets until the first win, including the winning ticket). Calculate the p.m.f. of X.
(b) Calculate the expectation $\mathbb{E}\, X$.
(c) Calculate the variance $\sigma^2(X)$.
(d) What are your expected winnings in this game?

(9) Prove the following claims. Here, X, Y are discrete random variables on the same sample space, and $a, b \in \mathbb{R}$.
(a) $\mathbb{E}\,(aX + b) = a\mathbb{E}\, X + b$
(b) $\sigma^2(aX + b) = a^2\sigma^2(X)$
(c) $\sigma^2(X) = \mathbb{E}(X^2) - (\mathbb{E}\, X)^2$

(10) Textbook Problem 3.5.

(11) Textbook problem 3.7.

(12) **Challenge, not marked.** In a town, there are on average 2.3 children in a family and a randomly chosen child has on average 1.6 siblings. Determine the variance of the number of children in a randomly chosen family.