Math 302, Assignment 1

(1) Let $S = \{1, \{\}, c\}$ be a sample space. List all possible events.

(2) Let Ω be a sample space and $\mathbb{P}$ be a probability. Prove that there can't exist events E, F that satisfy

$$\mathbb{P}(E \setminus F) = \frac{1}{3}, \ \mathbb{P}(E \cup F) = \frac{1}{2}, \text{ and } \mathbb{P}\left((E \cap F)^c\right) = \frac{3}{4}.$$

(3) We roll a fair die until the first 1 comes up. What is the probability that the number of tosses is odd?

(4) Assuming a fair poker deal, what is the probability of a
 (a) royal flush
 (b) straight flush
 (c) flush
 (d) straight
 (e) two pair

 See `https://en.wikipedia.org/wiki/List_of_poker_hands` for the definition of these poker hands.

(5) How many ways are there to deal 52 standard playing cards to four players (so that each player gets 13 cards)? Suppose you are world champion in card dealing, and can deal 52 cards in just one second. Compare the time it would need you to deal all possible combinations with the current age of the universe.

(6) We toss a fair die four times. What is the probability that all tosses produce different outcomes?

(7) **Challenge, not marked.** Prove that the number of unordered sequences of length k with elements from a set X of size n is $\binom{n+k-1}{k}$.
 Hint: For illustration, first consider the example $n = 4, k = 6$. Let the 4 elements of the set X be denoted a, b, c, d. Argue that any unordered sequence of size 6 consisting of elements a, b, c, d can be represented uniquely by a symbol similar to "$\cdot\cdot\,|\cdot|\cdot\cdot|\cdot$", corresponding to the sequence *aabccd*. Now count the number of choices for the vertical bars.

(8) You own n colors, and want to use them to color 6 objects. For each object, you randomly choose one of the colors. How large does n have to be so that odds are that no two objects will have the same color (i.e., every object is colored in a different color)?

(9) Assume that the events E_1, E_2 are independent.
 a) Prove that the events E_1^c, E_2^c are also independent.
 b) If, in addition, $\mathbb{P}(E_1) = \frac{1}{2}$ and $P(E_2) = \frac{1}{3}$, Prove that

$$\mathbb{P}(E_1 \cup E_2) = \frac{2}{3}.$$

 c) If, in addition, E_3 is a third event that is independent of E_1 and of E_2, and such that $\mathbb{P}(E_3) = \frac{1}{4}$. Prove that

$$\frac{17}{24} \leq \mathbb{P}(E_1 \cup E_2 \cup E_3) \leq \frac{19}{24}.$$

(10) Eight rooks are placed randomly on a chess board. What is the probability that none of the rooks can capture any of the other rooks? (In non-chess

terms: Randomly pick 8 unit squares from an 8×8 square grid. What is the probability that no two squares share a row or a column?)
Hint: How many choices do you have to place rooks in the first row? After you have made your choice, how many choices do you have for the second? Continue this reasoning.

(11) We toss two dice. Consider the events
E: The sum of the outcomes is even.
F: At least one outcome is 5.
Calculate the conditional probabilities $\mathbb{P}(E \mid F)$ and $\mathbb{P}(F \mid E)$.

(12) A fair die is rolled repeatedly.
(a) Give an expression for the probability that the first five rolls give a three at most two times.
(b) Calculate the probability that the first three does not appear before the fifth roll.
(c) Calculate the probability that the first three appears before the twentieth roll, but not before the fifth roll.

(13) Textbook problem 1.18.

(14) **Challenge, not marked.** Let the sequence of events $E_1, E_2, \ldots, E_n$ be independent, and assume that $\mathbb{P}(E_i) = \frac{1}{i+1}$. Show that $\mathbb{P}(E_1 \cup \cdots \cup E_n) = \frac{n}{n+1}$