Math 302, Assignment 3

1. Suppose that X has moment generating function

$$M_X(t) = \tfrac{1}{4}e^{-3t} + \tfrac{1}{2} + \tfrac{1}{4}e^t.$$

   (a) Find the mean and variance of X by differentiating the m.g.f. above.

   (b) Find the p.m.f. of X. Use your expression for the p.m.f. to check your answers from part (a).

2. You have two dice, one with three sides labeled $0, 1, 2$ and one with 4 sides, labeled $0, 1, 2, 3$. Let X_1 be the outcome of rolling the first die, and X_2 the outcome of rolling the second. The rolls are independent.

   (a) What is the joint p.m.f. of (X_1, X_2)?

   (b) Let $Y_1 = X_1 \cdot X_2$ and $Y_2 = \max\{X_1, X_2\}$. Make a table for the joint p.m.f. of (Y_1, Y_2).

   (c) Are Y_1, Y_2 independent?

3. Let $X \sim \mathrm{Exp}(2)$, $Y \sim \mathrm{Unif}([1, 3])$, and assume that X and Y are independent. Calculate $\mathbb{P}(Y - X \geq \frac{1}{2})$.

4. The random variables X, Y have joint probability density function

$$f(x, y) = \begin{cases} C\frac{e^{-x}-e^{-x-2y}}{e^y-1} & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) What is the value of C?

   (b) Are X and Y independent?

   (c) Find $\mathbb{P}(X < Y)$.

5. Let X_1 and X_2 be two discrete random variables with joint p.m.f. $\mathbb{P}(X_1 = k_1, X_2 = k_2)$. Prove the following claims from the lecture:

   (a) If $g : \mathbb{R}^2 \to \mathbb{R}$ is a function, then

$$\mathbb{E}\, g(X_1, X_2) = \sum_{k_1, k_2} g(k_1, k_2) \cdot \mathbb{P}(X_1 = k_1, X_2 = k_2).$$

   Hint: Remember that the left hand side is by definition $\mathbb{E}\, g(X_1, X_2) = \sum_l l \cdot \mathbb{P}(g(X_1, X_2) = l)$, where the sum is over all values of $g(X_1, X_2)$, i.e. over all l such that $l = g(k_1, k_2)$ for some value k_1 of X_1 and some value k_2 of X_2.

   (b) $\mathbb{E}[X_1 + X_2] = \mathbb{E}\, X_1 + \mathbb{E}\, X_2$. Hint: Use part (a).

6. Let X and Y be either two independent Poisson RV's, or two independent Exponential RV's, with parameters μ, λ. Compute the p.m.f. / p.d.f. of $X + Y$.

7. Compute the moment generating functions of the Geom(p) and the Exp(λ) random variables.

8. **Challenge, not marked** Let X be a continuous random variable with p.d.f. $f(x)$ and $g : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Show that the p.d.f. of $g(X)$ equals

$$f_{g(X)}(y) = \frac{f(g^{-1}(y))}{g'(g^{-1}(y))}$$

9. Textbook exercises 5.6 and 5.7.