The University of British Columbia

Final Examination - April 18, 2017

MATH 223

Time: 2.5 hours

Last Name ______________________________ **First** ________________________

Signature ____________________ **Student Number** ______________________

Special Instructions:

One 3 inch by 5 inch index card of notes allowed. No memory aids are allowed. No calculators may be used. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page and indicate that you have done so. Where boxes are provided for answers, put your final answers in them.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
Total		70

1. (10 pts) Find an orthonormal basis of eigenvectors for the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

Answer

2. Determinants!

(a) (3 pts) Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 3 & 7 \\ 0 & 0 & 4 & 5 & 6 & 4 \\ 0 & 2 & 3 & 4 & 5 & 11 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}.$$

Calculate $\det(A)$.

Answer

(b) (3 pts) Let $A \in \mathbb{R}^{3\times 3}$ be a diagonalizable matrix and suppose you are given the eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Is that enough information to determine $\det(A)$? If so, what is $\det(A)$?

Answer

(c) Let $A \in \mathbb{R}^{4\times 3}$ and suppose you are given the singular values $\sigma_1, \sigma_2, \sigma_3$.

i. (2 pts) Is that enough information to determine $\det(A^T A)$? If so, what is $\det(A^T A)$?

Answer

ii. (2 pts) Is that enough information to determine $\det(AA^T)$? If so, what is $\det(AA^T)$?

Answer

3. (10 pts)

Solve the differential equation given initial conditions $x_1(0) = 1, x_2(0) = 1$.

$$\frac{d}{dt}x_1(t) = x_1(t) + 2x_2(t)$$
$$\frac{d}{dt}x_2(t) = 2x_1(t) + 4x_2(t)$$

Answer

4. (10 pts) You are attempting to solve for x_1, x_2 in the matrix equation $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

There is no exact solution, so instead find a 'least squares' choice $\hat{b}$ in the column space of A (and hence with $||b - \hat{b}||$ being minimized) and solve the new system $Ax = \hat{b}$ for x_1, x_2. Include $\hat{b}, x_1$, and x_2 in the answer box. *Hint: Remember that there is a way to check your answer!*

Answer

5. Find the matrix!

(a) (5 pts) Let A be a **3 by 3** matrix of your choosing. From the orthogonality relation, $R(A) = N(A^T)^\perp$ we know that it is impossible for $R(A)$ to be equal to $N(A^T)$. However, is there some choice of A which satisfies $R(A) = N(A)$? If so, find one, and give its SVD (any form). If not, why not?

Answer

(b) (5 pts) Let A be a **4 by 4** matrix of you choosing. Is there some choice of A which satisfies $R(A) = N(A)$? If so, find one, and give its SVD (any form). If not, why not?

Answer

6. (10 pts) Consider the inner product space of degree-2 polynomials over $\mathbb{R}$: $V = \{a + bx + cx^2 : a, b, c \in \mathbb{R}\}$ with inner product

$$< f, g >:= \int_{-1}^{1} f(x)g(x)dx, \qquad f, g \in V.$$

This inner product induces a norm:

$$||f|| := \sqrt{< f, f >}.$$

Consider also the subspace of degree-1 polynomials $U = \{a + bx : a, b \in \mathbb{R}\}$.

Let $g(x) = 1 + x + x^2 \in V$.

Find the degree-1 polynomial, $f \in U$ which minimizes

$$||f(x) - g(x)||$$

over all $f \in U$.

Hint: Your first guess may be $f(x) = 1 + x$, but this is not correct.

Answer

7. (10 pts) We start this question by inductively building a large matrix. Let

$$H_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and let

$$H_{k+1} = \begin{bmatrix} H_k & H_k \\ H_k & -H_k \end{bmatrix} \in \mathbb{R}^{2^k \times 2^k}, \qquad k = 1, 2, 3, \ldots.$$

For example,

$$H_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Let $y = [1, 2, 3, \ldots, 2^{100}]^T \in \mathbb{R}^{2^{100}}$. Let x be the solution to

$$H_{100}\, x = y.$$

What is the value of the first entry of x?

Answer

The End