Math 223 Assignment 8

Question 2 is pretty likely to be similar to a question on the final. Please note the very important fact that an $n \times n$ symmetric matrix always is diagonalizable with an orthonormal basis of eigenvectors for $\mathbf{R}^n$. This will be covered in class but perhaps not completed before the due date of this assignment.

1. Find an orthonormal basis for $\mathbf{R}^3$ by applying Gram-Schmidt to the three vectors:

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Please recall that we can change a vector with fractional entries to one with integral entries by rescaling and we do not need to normalize until the last step.

2. Find orthonormal bases of eigenvectors for the following matrices (you can find such questions on every exam):

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & -2 & 1 \\ -2 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix}$$

Hint: for B, 0 is an eigenvalue, and for C, -2 is an eigenvalue.

3. (adapted from an exam) Let A be a 3×3 matrix with $\det(A - \lambda I) = -(\lambda^3 + a\lambda^2 + b\lambda + c)$. Show that if A is diagonalizable, then the following equation is true

$$A^3 + aA^2 + bA + cI = 0$$

(this equation is in fact true for any 3×3 matrix and is a special case of the Cayley-Hamilton Theorem).

4. Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be non-zero vectors satsifying $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for all pairs i, j with $i \neq j$. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ are linearly independent.

5. (from an exam) Let A be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (some may repeat) and an orthonormal basis of eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ ($A\mathbf{u}_i = \lambda_i \mathbf{u}_i$). Then show that

$$A = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{u}_i^T.$$

(thus A is a sum of n symmetric rank 1 matrices)

6. (from an exam) Consider a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We could attempt to solve for A^{-1} by letting

$$A^{-1} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

with four variables x, y, z, t and then $AA^{-1} = I$ becomes a system of 4 equations in 4 unknowns with an associated 4×4 matrix B. What is the rank of the 4×4 system of equations assuming A^{-1} exists? Explain. Can you say anything about the rank of B if $\det(A) = 0$? Explain.

7. (from an exam) You are attempting to solve for x, y, z in the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Find a 'least squares' choice $\hat{\mathbf{b}}$ in the column space of A (and hence with $||\mathbf{b} - \hat{\mathbf{b}}||^2$ being minimized) and then solve the new system $A\mathbf{x} = \hat{\mathbf{b}}$ for x, y, z. You can check your choice of $\hat{\mathbf{b}}$ by testing if $\mathbf{b} - \hat{\mathbf{b}}$ is orthogonal to the column space of A.