

1. Let  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  be a basis for a vector space  $V$ . Then if we define  $\mathbf{v}_1 = \mathbf{u}_1 + 2\mathbf{u}_3$ ,  $\mathbf{v}_2 = \mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_3$ ,  $\mathbf{v}_3 = \mathbf{u}_2 - \mathbf{u}_3$ , show that  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  forms a basis for  $V$ .
2. (from a test)

$$\text{Let } \mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

$$\text{NOTE: } \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 4 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

Let  $f : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  be the linear transformation satisfying

$$f(\mathbf{w}_1) = \mathbf{w}_2 - \mathbf{w}_3, \quad f(\mathbf{w}_2) = -\mathbf{w}_2 + \mathbf{w}_3, \quad f(\mathbf{w}_3) = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3.$$

- a) Give the matrix representation of  $f$  with respect to the basis  $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ .
- b) Give the matrix representation of  $f$  where the input  $\mathbf{x}$ , is written with respect to the basis  $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$  and the output  $f(\mathbf{x})$  is written with respect to the basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  (the standard basis).
- c) Is  $\mathbf{w}_1$  in the range of  $f$ ?
3. (from a test)

$$\text{Let } \mathbf{z}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{z}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{z}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{NOTE: } \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

Let  $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  be the linear transformation satisfying

$$T(\mathbf{z}_1) = 2\mathbf{z}_2, \quad T(\mathbf{z}_2) = 2\mathbf{z}_2, \quad T(\mathbf{z}_3) = \mathbf{z}_1 + \mathbf{z}_2.$$

- a) Give the matrix representation of  $T$  with respect to the basis  $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ .
- b) Give the matrix representation of  $T$  with respect to the basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  (the standard basis). Give the explicit matrix with integer entries.
- c) Give the matrix representing  $T^2$  with respect to the basis  $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ . What is the rank of the matrix representing  $T^2$  with respect to the standard basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ ?
4. Solve the system of differential equations

$$\begin{aligned} \frac{d}{dt}x_1(t) &= -5x_1(t) + 6x_2(t) \\ \frac{d}{dt}x_2(t) &= -1x_1(t) \end{aligned}$$

First find the general solution for  $x_1(t), x_2(t)$  as a function of  $x_1(0), x_2(0)$ . Given  $x_1(0) = 6$  and  $x_2(0) = 2$ , find the solutions explicitly and compute

$$\lim_{t \rightarrow \infty} \frac{x_1(t)}{x_2(t)}$$

5. Let  $U$  and  $V$  be two 5-dimensional subspaces of  $\mathbf{R}^9$ . Show that there is a nonzero vector in  $U \cap V$ , the intersection of  $U$  and  $V$ . You may find it helpful to consider the case of two 2-dimensional subspaces of  $\mathbf{R}^3$  first but you won't be able to use ideas of lines and planes in  $\mathbf{R}^9$ .
6. The following question (Part c) is from a Putnam exam. This version has some added hints (Parts a and b) to perhaps make it doable.
- a) Show that it is impossible to have vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{t}$  with

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{u}\mathbf{v}^T + \mathbf{w}\mathbf{t}^T$$

- b) We can encode polynomials in two variables  $x, y$  as a matrix so that the polynomial  $3x^2y - xy^2 + 5x^3y^3$  is encoded by

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Thus the  $i, j$  entry corresponds to the coefficient of  $x^{i-1}y^{j-1}$ . Say we have a polynomial  $p(x) = x + 2x^2$  and polynomial  $q(y) = 5 - y^2$  then we have  $p(x)q(y)$  encoded as

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & -1 & 0 \\ 10 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} [5 \ 0 \ -1 \ 0].$$

Show that in general that a product of two polynomials  $p(x), q(y)$  can be encoded as  $\mathbf{u}\mathbf{v}^T$  where  $\mathbf{u}$  encodes  $p(x)$  and  $\mathbf{v}$  encodes  $q(y)$ .

- c) Can there exist polynomials  $p(x), q(y), r(x), s(y)$  (each of maximum degree 3) such that  $p(x)q(y) + r(x)s(y) = 1 + xy + x^2y^2$ ?