1. Let $A_1 = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$. For each of these two matrices, determine the eigenvalues and for each eigenvalue determine an eigenvector. For A_2 the eigenvalues are a little more complicated making the computations a little harder. Then give the *diagonalization* of each matrix; namely an invertible matrix M and a diagonal matrix D with $AM = MD$. (The equation $AM = MD$ is important because it will yield $A = MDM^{-1}$ and $M^{-1}AM = D$).

2. Let A be a 2×2 matrix with two different eigenvalues λ_1, λ_2 and associated eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. Let $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$. Assume that $|\lambda_1| > |\lambda_2|$. Show that

$$\lim_{n\to\infty} \frac{A^n\mathbf{v}}{\lambda_1^n} = a\mathbf{v}_1.$$

How do you define the limit? For $a \neq 0$ this means that we see the eigenevector $\mathbf{v}_1$ appearing in the limit.

3. Review the notes on Fibonacci numbers. Let f_1, f_2 be two arbitrary integers, not both zero. Consider the sequence $f_1, f_2, f_3, f_4, \ldots$ where $f_i = f_{i-1} + f_{i-2}$ for $i = 3, 4, 5, \ldots$. We wish to show that

$$\lim_{n\to\infty} \frac{f_n}{f_{n-1}} = \frac{1+\sqrt{5}}{2}.$$

Firstly, explain why we can solve for c_1, c_2 in the vector equation

$$\begin{bmatrix} f_2 \\ f_1 \end{bmatrix} = c_1 \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$

Using our hypothesis that f_1, f_2 are not both zero, we deduce that c_1, c_2 are not both zero. Secondly, use our hypothesis that f_1, f_2 are integers, not both zero, to deduce $c_1 \neq 0$. The irrationality of $\sqrt{5}$ (which you need not prove) combined with c_1, c_2 being integers is important. Thirdly verify the limit. If you can't show $c_1 \neq 0$ then you can still proceed assuming $c_1 \neq 0$ to establish this limit.

Hint: use ideas of the previous question.

4. In this question, we explore the behaviour of A^n when A does not have distinct eigenvectors (up to rescaling).

   (a) Let

$$A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

   .

   i. Find all eigenvectors and eigenvalues of A.
   ii. Give a simple expression for A^n.

   (b) Consider a set of 2-tuples satisfying

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \qquad n = 0, 1, 2, \ldots$$

   Let $x_0 = y_0 = 1$. Give a relatively simple expression for $x_n + y_n$.

5. I wish to see the solutions to a system of equations in *Parametric Vector Form* (or *Vector Parametric Form*). For eample if the set of solutions is:

$$\begin{array}{rcllll} x_1 & = & & -3r & -4s & -2t \\ x_2 & = & & r & & \\ x_3 & = & & & -2s & \\ x_4 & = & & & s & \\ x_5 & = & & & & t \\ x_6 & = & 1/3 & & & \end{array}$$

for all choices $r, s, t \in \mathbf{R}$ then we can write the set of solutions in parametric vector form as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad r, s, t \in \mathbf{R}.$$

Give the vector parametric form of all solutions to the following system of equations:

$$\begin{array}{llllcl} 2x_1 & & +4x_4 & +6x_5 & = & 14 \\ 2x_1 & & +5x_4 & +7x_5 & = & 16 \\ 3x_1 & +2x_2 & +8x_4 & +9x_5 & = & 27 \\ 3x_1 & +4x_2 & +13x_4 & +12x_5 & = & 39 \end{array}$$

6. Give the solutions in vector parametric form for the plane $\pi = \{(x, y, z) : 2x - 2y + 3z = 5\}$.

7. Express the inverse of the following matrix A as a product of elementary matrices.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$