MATH 223: White and Blue Coordinates.

We intially understood that a vector $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ was given in the standard way with $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = a\cdot\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] + b\cdot\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and hence a units to the left and b units up from the origin.

We had an assignment question on assignment 1 changing between variables s, t and x, y. Our bird population model is an excellent example of changing coordinates. We have our standard coordinate system given by the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus the vector $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = a\cdot\mathbf{e}_1 + b\cdot\mathbf{e}_2$. We consider a and b as the *white* coordinates of the vector although of course it appears black. Think chalk.

We have the blue coordinate system given by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

Hopefully your viewer lets you see the colour difference. We discover that any vector can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & -4 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4/17 & 1/17 \\ 5/17 & -3/17 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We think of $A = [\mathbf{v}_1\, \mathbf{v}_2]$ as the matrix taking us from blue coordinates to white coordinates and A^{-1} necessarily takes from white coordinates to blue coordinates. We write this

$$\underset{\text{white} \leftarrow \text{blue}}{\begin{bmatrix} 3 & 1 \\ 5 & -4 \end{bmatrix}}, \qquad \underset{\text{blue} \leftarrow \text{white}}{\begin{bmatrix} 4/17 & 1/17 \\ 5/17 & -3/17 \end{bmatrix}}.$$

Some easy calculations have

$$\begin{bmatrix} 3 & 1 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 4/17 & 1/17 \\ 5/17 & -3/17 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 11/17 \\ 1/17 \end{bmatrix}.$$

Thus $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ in blue coordinates is the vector $\left[\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}\right]$ in white coordinates. Similarly $\left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right]$ in white coordinates is the vector $\left[\begin{smallmatrix} 11/17 \\ 1/17 \end{smallmatrix}\right]$ in blue coordinates.

Now we had been considering the linear transformation $f : \mathbf{R}^2 \to \mathbf{R}^2$ with $f(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}$$

Our application with diagonalization is the matrix equation

$$\underset{\substack{\text{white} \leftarrow \text{white} \\ f}}{\begin{bmatrix} .7 & .3 \\ 2 & 0 \end{bmatrix}} = \underset{\text{white} \leftarrow \text{blue}}{\begin{bmatrix} 3 & 1 \\ 5 & -4 \end{bmatrix}} \underset{\substack{\text{blue} \leftarrow \text{blue} \\ f}}{\begin{bmatrix} 1.2 & 0 \\ 0 & -.5 \end{bmatrix}} \underset{\text{blue} \leftarrow \text{white}}{\begin{bmatrix} 4/17 & 1/17 \\ 5/17 & -3/17 \end{bmatrix}}.$$

Thus in blue coordinates our linear transormation can be interpreted as a ‘simple’ diagonal matrix.

I wanted to include a picture of the two coordinate systems overlayed one over the other. The fine black lines are the integer gridlines of the white coordinates.