An $n \times n$ matrix $Q$ is *orthogonal* if $Q^T = Q^{-1}$. The columns of $Q$ would form an orthonormal basis for $\mathbf{R}^n$. The rows would also form an orthonormal basis for $\mathbf{R}^n$.

A matrix $A$ is *symmetric* if $A^T = A$.

**Theorem 0.1** *Let $A$ be a symmetric $n \times n$ matrix of real entries. Then there is an orthogonal matrix $Q$ and a diagonal matrix $D$ so that*

$$AQ = QD, \qquad i.e.\ Q^T AQ = D.$$

*Note that the entries of $M$ and $D$ are real.*

There are various consequences to this result:
A symmetric matrix $A$ is diagonalizable
A symmetric matrix $A$ has an othonormal basis of eigenvectors.
A symmetric matrix $A$ has real eigenvalues.

**Proof:** The proof begins with an appeal to the fundamental theorem of algebra applied to $\det(A - \lambda I)$ which asserts that the polynomial factors into linear factors and one of which yields an eigenvalue $\lambda$ *which may not be real.*

Our second step it to show $\lambda$ is real. Let $\mathbf{x}$ be an eigenvector for $\lambda$ so that $A\mathbf{x} = \lambda\mathbf{x}$. Again, if $\lambda$ is not real we must allow for the possibility that $\mathbf{x}$ is not a real vector.

Let $\mathbf{x}^H = \overline{\mathbf{x}}^T$ denote the conjugate transpose. It applies to matrices as $A^H = \overline{A}^T$. Now $\mathbf{x}^H\mathbf{x} \geq 0$ with $\mathbf{x}^H\mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$. We compute $\mathbf{x}^H A\mathbf{x} = \mathbf{x}^H(\lambda\mathbf{x}) = \lambda\mathbf{x}^H\mathbf{x}$. Now taking complex conjugates and transpose $(\mathbf{x}^H A\mathbf{x})^H = \mathbf{x}^H A^H\mathbf{x}$ using that $(\mathbf{x}^H)^H = \mathbf{x}$. Then $(\mathbf{x}^H A\mathbf{x})^H = \mathbf{x}^H A\mathbf{x} = \lambda\mathbf{x}^H\mathbf{x}$ using $A^H = A$. Important to use our hypothesis that $A$ is symmetric. But also $(\mathbf{x}^H A\mathbf{x})^H = \overline{\lambda}\mathbf{x}^H\mathbf{x} = \overline{\lambda}\mathbf{x}^H\mathbf{x}$ (using $\mathbf{x}^H\mathbf{x} \in \mathbf{R}$). Knowing that $\mathbf{x}^H\mathbf{x} > 0$ (since $\mathbf{x} \neq \mathbf{0}$) we deduce that $\lambda = \overline{\lambda}$ and so we deduce that $\lambda \in \mathbf{R}$.

The rest of the proof uses induction on $n$. The result is easy for $n = 1$ ($Q = [1]$!). Assume we have a real eigenvalue $\lambda_1$ and a real eigenvector $\mathbf{x}_1$ with $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ and $||\mathbf{x}_1|| = 1$. We can extend $\mathbf{x}_1$ to an orthonormal basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$. Let $M = [\mathbf{x}_1\, \mathbf{x}_2 \cdots \mathbf{x}_n]$ be the matrix formed with columns $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$. Then

$$AM = M\begin{bmatrix} \lambda_1 & B \\ \mathbf{0} & C \end{bmatrix} \text{ or } M^{-1}AM = \begin{bmatrix} \lambda_1 & B \\ \mathbf{0} & C \end{bmatrix}.$$

which is the sort of result from our assignments. But the matrix on the right is symmetric since it is equal to $M^{-1}AM = M^TAM$ (since the basis was orthonormal) and we note $(M^TAM)^T = M^TAM$ (using $A^T = A$ since $A$ is symmetric). Then $B$ is a $1 \times (n-1)$ zero matrix and $C$ is a symmetric $(n-1) \times (n-1)$ matrix.

By induction there exists an orthogonal matrix $N$ (with $N^T = N^{-1}$) and a diagonal matrix $E$ with $N^{-1}CN = E$. We form a new orthognal matrix

$$P = \begin{bmatrix} 1 & 0\,0 \cdots 0 \\ \mathbf{0} & N \end{bmatrix}$$

which has

$$P^{-1}\begin{bmatrix} \lambda_1 & B \\ \mathbf{0} & C \end{bmatrix}P = \begin{bmatrix} \lambda_1 & 0\,0 \cdots 0 \\ \mathbf{0} & E \end{bmatrix}$$

This becomes

$$P^{-1}M^{-1}AMP = \begin{bmatrix} \lambda_1 & 0\,0\cdots 0 \\ \mathbf{0} & E \end{bmatrix}$$

which is a diagonal matrix $D$. We note that $(MP)^T = P^TM^T = P^{-1}M^{-1}$ and so $Q = MP$ is an orthogonal matrix with $Q^TAQ = D$. This proves the result by induction. ■

Recall that for a complex number $z = a + bi$, the conjugate $\overline{z} = a - bi$. We may extend the conjugate to vectors and matrices. When we consider extending inner products to $\mathbf{C}^n$ we must define

$$< \mathbf{x}, \mathbf{y} >= \overline{\mathbf{x}}^T\mathbf{y}$$

so that $< \mathbf{x}, \mathbf{x} >\in \mathbf{R}$ and $< \mathbf{x}, \mathbf{x} >\geq 0$. Also $< \mathbf{y}, \mathbf{x} >= \overline{< \mathbf{x}, \mathbf{y} >}$. We would like some notation for the conjugate transpose. Some use the notation $A^H = (\bar{A})^T$ and $\mathbf{v}^H = (\bar{\mathbf{v}})^T$. Sometimes a dagger is used. We write the complex inner product $< \mathbf{v}, \mathbf{u} >= \mathbf{v}^H\mathbf{u}$.

A matrix $A$ is *hermitian* if $\overline{A}^T = A$. For example any symmetric matrix of real entries is also hermitian. The follow matrix is hermitian:

$$\begin{bmatrix} 3 & 1-2i \\ 1+2i & 4 \end{bmatrix}$$

One has interesting identities such as $< \mathbf{x}, A\mathbf{y} >=< A\mathbf{x}, \mathbf{y} >$ when $A$ is hermitian.

**Theorem** Let $A$ be a hermitian matrix. Then there is a unitary matrix $M$ with entries in $\mathbf{C}$ and a diagonal matrix $D$ of real entries so that

$$AM = MD, \qquad A = MDM^{-1}$$

As an example let

$$A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

We compute

$$\det(A - \lambda I) = \begin{bmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda$$

and thus the eigenvalues are $0, 2$ (Note that they are real which is a consequence of the theorem). We find that the eigenvectors are

$$\lambda_1 = 2 \quad \mathbf{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad \lambda_2 = 0 \quad \mathbf{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Not surprisingly $< \mathbf{v}_1, \mathbf{v}_2 >= 0$, another consequence of the theorem. We would have to make them of unit length to obtain an orthonormal basis:

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \qquad AU = UD$$

Let $A$ be an $n \times n$ matrix. The proof of either of our theorems first requires us to find a real eigenvalue. Assume $A$ is hermitian so that $A^H = A$. Now $\det(A - \lambda I)$ is a polynomial in $\lambda$ of degree $n$. By the Fundamental theorem of Algebra it factors into linear factors. Let $\mu$ be a root which might not be real but will be complex. Let $\mathbf{v}$ be an eigenvector of eigenvalue $\mu$ (computed using our standard Gaussian Elimination over $\mathbf{C}$). Then $A\mathbf{v} = \mu\mathbf{v}$. We compute $\mathbf{v}^HA^H = \mu\mathbf{v}^H$. By symmetry of $A$, $\mathbf{v}^TA^T = \mathbf{v}^TA$. Thus $\mathbf{v}^TA\mathbf{v} = \mu\mathbf{v}^H\mathbf{v}$ but also

An $n \times n$ matrix $U$ is *unitary* if $\overline{U}^T = U^{-1}$. The columns of $U$ would form an orthonormal basis for $\mathbf{C}^n$. The rows would also form an orthonormal basis for $\mathbf{C}^n$. The following matrix is unitary:

$$\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

since $\overline{\begin{bmatrix} 1 \\ i \end{bmatrix}} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -i \end{bmatrix}^T \begin{bmatrix} 1 \\ -i \end{bmatrix} = 0$.

Using this inner product one can perform Gram Schmidt on complex vectors (but be careful with the order since in general $< \mathbf{u}, \mathbf{v} > \neq < \mathbf{v}, \mathbf{u}.$):

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} i \\ 1+i \end{bmatrix}, \quad < \mathbf{v}_1, \mathbf{v}_2 >= [2 \;\; 1-i] \begin{bmatrix} i \\ 1+i \end{bmatrix} = 2i + 2.$$

$$\mathbf{u}_1 = \mathbf{v}_1, \quad \mathbf{u}_2 = \mathbf{v}_2 - \frac{< \mathbf{u}_1, \mathbf{v}_2 >}{< \mathbf{u}_1, \mathbf{u}_1 >} \mathbf{u}_1 = \begin{bmatrix} i \\ 1+i \end{bmatrix} - \frac{2+2i}{6} \begin{bmatrix} 2 \\ 1+i \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} + \frac{1}{3}i \\ 1 + \frac{1}{3}i \end{bmatrix}$$

You may check

$$< \mathbf{u}_1, \mathbf{u}_2 >= [2 \;\; 1+i] \begin{bmatrix} -\frac{2}{3} + \frac{1}{3}i \\ 1 + \frac{1}{3}i \end{bmatrix} = -\frac{4}{3} + \frac{2}{3}i + \frac{4}{3} - \frac{2}{3}i = 0$$

To form a unitary matrix we must normalize the vectors.

$$\begin{bmatrix} 2 \\ 1+i \end{bmatrix} \to \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}}i \end{bmatrix}, \quad \begin{bmatrix} -\frac{2}{3} + \frac{1}{3}i \\ 1 + \frac{1}{3}i \end{bmatrix} \to \begin{bmatrix} -2+i \\ 3+i \end{bmatrix} \to \begin{bmatrix} -\frac{2}{\sqrt{15}} + \frac{1}{\sqrt{15}}i \\ \frac{3}{\sqrt{15}} + \frac{1}{\sqrt{15}}i \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{2}{\sqrt{15}} + \frac{1}{\sqrt{15}}i \\ \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}}i & \frac{3}{\sqrt{15}} + \frac{1}{\sqrt{15}}i \end{bmatrix}$$

where we can check $\overline{U}^T U = I$. Best to let a computer do these calculations!