MATH 223: Some results for 2×2 matrices.
Richard Anstee

Multiplicative Inverses

It would be nice to have a multiplicative *inverse*. That is given a matrix A, find the inverse matrix A^{-1} so that $AA^{-1} = A^{-1}A = I$. Such an inverse can be shown to be unique, if it exists (How?).

The following remarkable fact is useful where we introduce A^*, known as the *adjoint* of A,:

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \underset{A^*}{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \det(A)I$$

where we have defined

$$\det(A) = \det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc.$$

Now if $\det(A) \neq 0$, then

$$A \cdot (\frac{1}{\det(A)}A^*) = I$$

and so it is sensible to define

$$A^{-1} = \frac{1}{\det(A)}A^*$$

and we find that $AA^{-1} = I$ and then we can verify that $A^{-1}A = I$ as well so that A^{-1} is the multiplicative inverse of A. One verification is obtained by showing $A^*A = \det(A)I$.

If $\det(A) \neq 0$, then A has an inverse A^{-1} of the form

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

If $\det(A) = 0$, then we can show no inverse exists. If $A = 0$, then we can easily verify that $AB = 0$ for any choice of B and so there can be no A^{-1}. If $A \neq 0$, we note that $AA^* = 0$ and we get a contradiction by computing

$$A^* = A^{-1}AA^* = A^{-1}0 = 0.$$

A better way to state this is as follows: If $\det(A) = 0$, then there exists an $\mathbf{x} \neq \mathbf{0}$ with $A\mathbf{x} = \mathbf{0}$ and hence A^{-1} does not exist. The choice of $\mathbf{x}$ could either be a non zero column of A^* or in the event that A^* is 0, then any non zero vector $\mathbf{x}$ would do. We compute to get a contradiction as before:

$$\mathbf{x} = I\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}.$$

Another approach is to note that A has an inverse if and only if the two columns of A are not multiples of one another. This is the observation that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has } ad - bc \neq 0 \text{ if}$$

$$\text{the fractions } \frac{a}{c} \neq \frac{b}{d} \text{ and so } \begin{bmatrix} a \\ c \end{bmatrix} \neq k \begin{bmatrix} b \\ d \end{bmatrix} \text{ for any } k.$$

Of course, this argument must be extended to take care of cases where either $c = 0$ or $d = 0$, but I will leave that as an exercise.

We can check that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Rather more remarkably, we find

$$\det(AB) = \det(A)\det(B)$$

which we can verify using arbitrary matrices.

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

We compute

$$\det(A)\det(B) = (ad-bc)(eh-gf) = adeh - adgf - bceh + bcgf$$

$$\det(AB) = (ae+bg)(cf+dh) - (af+bh)(ce+dg) =$$
$$acef + adeh + bcfg + bdgh - acef - adfg - bceh - bdgh.$$

Noting the remarkable cancellation of the terms terms $acef$ and $bdgh$, we verify the equality $\det(AB) = \det(A)\det(B)$. (*Aside: a general proof for larger matrices will have a different flavour, this particular proof can also be generalized*)