MATH 223: Proof that row interchanges change sign of determinants.

We use the notation $E(i,j)$ to denote the elementary row operation of interchanging row i and row j. To prove that

$$\det(E(i,j)A) = -\det(A)$$

it suffices to prove that

$$\det(E(1,2)A) = -\det(A).$$

The book attempts to do this by induction but then assumes that they can use expansion about any row. The natural way to obtain the expansion formulas about any row is to use the above result. The book's argument is then circular as given (although they would be able to recover if they had an independent way to verify that expansion about any row follows the standard formula). For convenience, use the notation M_{ij} to denote the matrix obtained from A by deleting row i and column j. We need to run our recursive definition for det by first expanding about the first row and then for each determinant of M_{ij}, an $(n-1)\times(n-1)$ matrix, expand again about its first row. Thus let $M_{ij,kl}$ to denote the matrix obtained from M_{ij} by deleting row k (of A not M_{ij}) and column l (of A not M_{ij}). This will become complicated as we try to determine the appropriate power of (-1) to use.

Now we expand the two determinants.

$$\det(A) = (-1)^{1+1}a_{11}\det M_{11} + (-1)^{1+2}a_{12}\det M_{12} + \cdots + (-1)^{1+n}a_{1n}\det M_{1n}$$

$$= (-1)^{1+1}a_{11}\Big((-1)^{1+1}a_{22}\det M_{11,22} + (-1)^{1+2}a_{23}\det M_{11,23} + \cdots + (-1)^{1+(n-1)}a_{2n}\det M_{11.2n}\Big)$$

$$+(-1)^{1+2}a_{12}\Big((-1)^{1+1}a_{21}\det M_{12,21} + (-1)^{1+2}a_{23}\det M_{12,23} + \cdots + (-1)^{1+(n-1)}a_{2n}\det M_{12.2n}\Big)$$

$$+(-1)^{1+n}a_{1n}\Big((-1)^{1+1}a_{21}\det M_{1n,21} + (-1)^{1+2}a_{22}\det M_{1n,22} + \cdots + (-1)^{1+(n-1)}a_{2n}\det M_{1n.2(n-1)}\Big)$$

$$= \sum_{1\le i<j\le n} (-1)^{2+i+j-1}(a_{1i}a_{2j} - a_{1j}a_{2i})\det M_{1i,2j}.$$

The last formula is difficult to see. We need only consider products of the form $a_{1i}a_{2j}$ or $a_{1j}a_{2i}$ for a fixed pair i,j with $i<j$. The associated matrix minor is either $M_{1i,2j}$ or $M_{1j,2i}$ (but of course $M_{1i,2j} = M_{1j,2i}$ as matrices). So the remaining problem is the sign. The term with a_{1i} from the first expansion will have a sign $(-1)^{1+i}$. The subsequent term from expansion about the second row with a_{2j} has a sign $(-1)^{1+(j-1)}$ since column $j-1$ of M_{1i} is column j of A. This yields a sign term $(-1)^{2+i+j-1}$. The term with a_{1j} from the first expansion will have a sign $(-1)^{1+j}$. The subsequent term from expansion about the second row with a_{2i} has a sign $(-1)^{1+i}$ since column i of M_{1j} is column i of A (we are using $i<j$). This yields a sign term of $(-1)^{2+i+j} = -(-1)^{2+i+j-1}$. These two arguments have verified the formula above.

We perform the same task for $E(1,2)A$ where we let $E(1,2)A = (a'_{ij})$ i.e. a'_{ij} is the (i,j) entry of the matrix $E(1,2)A$. Let M'_{ij} denote the matrix obtained from $E(1,2)A$ by deleting its row i and its column j. We define $M'_{ij,kl}$ as the matrix obtained from M'_{ij} by deleting row k (of $E(1,2)A$ not M'_{ij}) and column l (of $E(1,2)A$ not M_{ij}). Given that $E(1,2)A$ is the same as A apart from the first two rows, we see that $M_{1i,2j} = M'_{1i,2j}$. Now we can copy our work from $\det(A)$:

$$\det(E(1,2)A) = (-1)^{1+1}a'_{11}\det M'_{11} + (-1)^{1+2}a'_{12}\det M'_{12} + \cdots + (-1)^{1+n}a'_{2n}\det M'_{1n}$$

$$= \sum_{1\le i<j\le n} (-1)^{2+i+j-1}(a'_{1i}a'_{2j} - a'_{1j}a'_{2i})\det M'_{1i,2j}$$

$$= \sum_{1\le i<j\le n} (-1)^{2+i+j-1}(a_{2i}a_{1j} - a_{2j}a_{1i})\det M_{1i,2j}$$

(using $a'_{1p} = a_{2p}$, $a'_{2q} = a_{1q}$ and $M'_{1p,2q} = M_{1p,2q}$)

$$= -\det(A).$$

I find this argument a bit delicate, but not difficult. We can use summation signs to simplify the writing in the argument above.

$$\det(A) = \sum_{i=1}^{n}(-1)^{1+i}a_{1i}\det M_{1i}$$

$$= \sum_{i=1}^{n}(-1)^{1+i}a_{1i}\Big(\sum_{\substack{j=1\\ j\neq i}}^{n}(-1)^{1+j-\delta}a_{2j}\det M_{1i,2j}\Big)$$

where

$$\delta = \begin{cases} 1 & \text{if } j>i \\ 0 & \text{if } j<i \end{cases}.$$

The expression for δ is the same count that was used above e.g when $j > i$, we must subtract one because column $j-1$ of M_{1i} is from column j of A.

Similarly, applying expansion about the first row of $E(1,2)A$ (which is the second row of A), and then expansion about the first rows of the Minors (which will come from the first row of A):

$$\det(E(1,2)A) = \sum_{j=1}^{n}(-1)^{1+j}a_{2j}\det M_{2j}$$

$$= \sum_{j=1}^{n}(-1)^{1+j}a_{2j}\Big(\sum_{\substack{i=1\\ i\neq j}}^{n}(-1)^{1+i-\delta}a_{1i}\det M_{2j,1i}\Big)$$

where

$$\delta = \begin{cases} 1 & \text{if } i>j \\ 0 & \text{if } i<j \end{cases}.$$

We deduce $\det(A) = -\det(E(1,2)A)$.