If we have a set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ where we set $U = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$, it is natural to express any vector $u \in U$ as a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$, namely

$$\mathbf{u} = c_1\mathbf{u}_2 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$$

where we think of $c_1, c_2, \ldots, c_k$ as the *coordinates* of $\mathbf{u}$ with respect to the spanning set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$. Now if $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly independent, then the coordinates behave as we would hope, namely they are unique.

Theorem 1 *If the set* $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ *is linearly independent, then for each vector* $\mathbf{u} \in U = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$*, there are unique numbers* $c_1, c_2, \ldots, c_k$ *(the coordinates) such that* $\mathbf{u} = c_1\mathbf{u}_2 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$.

Proof: The existence of numbers $c_1, c_2, \ldots, c_k$ follows from the fact that $\mathbf{u} \in U = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$. Assume

$$\mathbf{u} = c_1\mathbf{u}_2 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$$
$$\mathbf{u} = d_1\mathbf{u}_2 + d_2\mathbf{u}_2 + \cdots + d_k\mathbf{u}_k$$

Then by subtracting the two equations we obtain

$$\mathbf{0} = (c_1 - d_1)\mathbf{u}_2 + (c_2 - d_2)\mathbf{u}_2 + \cdots + (c_k - d_k)\mathbf{u}_k.$$

Since the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly independent, then we deduce that $c_1 - d_1 = 0$, $c_2 - d_2 = 0$, …, $c_k - d_k = 0$ and hence $c_1 = d_1$, $c_2 = d_2$, … ,$c_k = d_k$. ■

Thus if we have a k-dimensional vector space than we can coordinatize the vectors as elements of $\mathbf{R}^k$. Consider the following 4 vectors.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$$

We can verify that $U = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3.\mathbf{v}_4\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ noting that $\mathbf{v}_3 = -7\mathbf{v}_1 + 4\mathbf{v}_2$ and $\mathbf{v}_4 = -5\mathbf{v}_1 + 4\mathbf{v}_2$. Indeed $\dim(U) = 2$. While $U \subseteq \mathbf{R}^3$ it is natural to consider U as a 2-dimensional vector space and in fact we can write our vectors in blue coordinates with respect to the basis $\mathbf{v}_1, \mathbf{v}_2$ of U.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ is } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \text{ is } \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} \text{ is } \begin{bmatrix} -7 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \text{ is } \begin{bmatrix} -5 \\ 4 \end{bmatrix}.$$

A somewhat different example is from the assignment. Let $W = \text{span}\{\cos^2(x), \sin^2(x)\}$. We deduce that $\{\cos^2(x), \sin^2(x)\}$ is a basis for W so we can coordinatize with respect to this basis.

$$\cos^2(x) \text{ is } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \sin^2(x) \text{ is } \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad 2 \text{ is } \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \cos(2x) \text{ is } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

As a vector space over $\mathbf{R}$ we can think of W as $\mathbf{R}^2$. Of course as functions, there are more properties. We can't differentiate a vector but we can differentiate $\cos^2(x)$.

A student in MATH 223 in 2015 said that U and W were *thinly veiled* examples of $\mathbf{R}^2$. And of course similarly we think of a vector space X with $\dim(X) = k$ as a *thinly veiled* example of $\mathbf{R}^k$. To make this precise consider the following definition.

Definition 2 *Given two vector spaces U, V over the same field F, we say that U and V are isomorphic if there is a bijective map $h : U \to V$ with $h(\mathbf{0}) = \mathbf{0}$ (the first $\mathbf{0}$ is in U and the second $\mathbf{0}$ is in V) and with the property that for any $\mathbf{x}, \mathbf{y} \in U$, we have $h(\mathbf{x}+\mathbf{y}) = h(\mathbf{x}) + h(\mathbf{y})$ and for any $c \in F$, $h(c\mathbf{x}) = ch(x)$.*

Remember that the isomorhism need not preserve other properties of the elements of U and V that are not associated with being a vector space.

Theorem 3 *If U and V are vector spaces over the same field and $\dim(U) = \dim(V)$ then U and V are isomorphic.*

Proof: Let $k = \dim(U) = \dim(V)$. Assume $k > 0$. Let U have basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ and V has basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Then define $h(\mathbf{u}_i) = \mathbf{v}_i$ and extend to all vectors of U by linearity; namely for $\mathbf{u} = \sum_{i=1}^k a_i\mathbf{u}_i$ and so define $h(\mathbf{u}) = \sum_{i=1}^k a_i\mathbf{v}_i$. We easily show that h is a bijection and $h^{-1}(\mathbf{v}_i) = \mathbf{u}_i$.

When $0 = \dim(U) = \dim(V)$, then each consists of just the zero vector and so the isomorphism is easy. ■

The following is an important application of dimension.

Theorem 4 *An $m \times m$ matrix A is diagonalizable if and only if there is a basis of $\mathbf{R}^m$ consisting of eigenvectors of A.*

Proof: If A is diagonalizable then there is a diagonal matrix D and an invertible matrix M with $AM = MD$. But then each column of M is an eigenvector of A (no column of M can be $\mathbf{0}$ since M is invertible. And since M is invertible, the columns of M are linearly independent and since there are m of them they form a basis for $\mathbf{R}^m$.

If there is a basis of $\mathbf{R}^m$ say $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ then if we form the matrix M whose columns are the $\mathbf{v}_i$'s then M is invertible. If $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$, then we have $AM = MD$ where the ith diagonal entry is λ_i. ■

Some examples. Imagine we have a 3-dimensional vector space $V = \text{span}\{f_1(x), f_2(x), f_3(x)\}$ where $f_1(x) = e^x$, $f_2(x) = e^{2x}$ and $f_3(x) = e^{3x}$. Demonstrating that these three are linearly independent is relatively easy (you could even examine the differing growth rates of the functions to prove linear independence). We can think of $\{f_i(x), f_2(x), f_3(x)\}$ as a basis F for V. We consider the linear transformation $T : V \to V$ defined as

$$T(h(x)) = h(x) + \frac{d}{dt}h(x).$$

We can represent T by a matrix when considering vectors in V written with respect to F.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

T with respect to F

We can consider other coordinate systems for V. Let $g_1(x) = e^x + e^{2x}$, $g_2(x) = e^{2x} + e^{3x}$ and $g_3(x) = e^x + e^{3x}$. We have the following

$$M = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{ccc} g_1 & g_2 & g_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$F \leftarrow G$

We can compute

$$M^{-1} = \begin{array}{c} g_1 \\ g_2 \\ g_3 \end{array} \overset{\begin{array}{ccc} f_1 & f_2 & f_3 \end{array}}{\underset{G \leftarrow F}{\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}}}$$

The existence of M^{-1} means that $f_1, f_2, f_3 \in \text{span}\{g_1(x), g_2(x), g_3(x)\}$ and easily we see $\text{span}\{g_1(x), g_2(x), g_3(x)\} \subseteq V$ from which we deduce that $\text{span}\{g_1(x), g_2(x), g_3(x)\} = V$ and so $\{g_1(x), g_2(x), g_3(x)\}$ forms a basis for V. What is T written as a matrix with respect to G?

$$\underset{G \leftarrow F}{\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}} \underset{T \text{ with respect to } F}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}} \underset{F \leftarrow G}{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} = \underset{T \text{ with respect to } G}{\begin{bmatrix} 5/2 & -1/2 & -1 \\ 1/2 & 7/2 & 1 \\ -1/2 & 1/2 & 3 \end{bmatrix}}$$

You can check

$$T(g_1 + g_2) = T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_G\right) = \underset{T \text{ with respect to } G}{\begin{bmatrix} 5/2 & -1/2 & -1 \\ 1/2 & 7/2 & 1 \\ -1/2 & 1/2 & 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_G = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}_G \qquad (1)$$

We note that $g_1(x) + g_2(x) = e^x + 2e^{2x} + e^{3x} = f_1(x) + 2f_2(x) + f_3(x)$ so that

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_G = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}_F$$

We $T(f_1(x) + 2f_2(x) + f_3(x))$ is computed as

$$\underset{T \text{ with respect to } F}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}_F = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}_F = 2f_1(x) + 6f_2(x) + 4f_3(x).$$

We compute $2f_1(x)+6f_2(x)+4f_3(x) = 2e^x+6e^{2x}+4e^{3x} = 2(e^x+e^{2x})+4(e^{2x}+e^{3x}) = 2g_1(x)+4g_2(x)$. This is (1) above.