Q: Approximate $(998)^{1/3}$ without calculator.

Let $f(x) = x^{1/3}$. We want $f(998)$.

Note: $f(1000) = 10$.

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$$

Linear approx at $x = 1000$:

$$L(x) = f(1000) + f'(1000)\,(x-1000)$$
$$= 10 + \frac{1}{3}(1000)^{-\frac{2}{3}}(x-1000)$$
$$= 10 + \frac{1}{3}\cdot\frac{1}{100}(x-1000)$$

$$\Rightarrow L(998) = 10 + \frac{1}{300}\cdot(998-1000) = \boxed{10 - \frac{2}{300}}$$

Over-estimate or under-estimate?

Note: $f''(x) = \frac{1}{3}\cdot\left(-\frac{2}{3}\right)x^{-\frac{5}{3}} = -\frac{2}{9}x^{-\frac{5}{3}} < 0$ for $x > 0$

$\Rightarrow$ f is concave down for $x > 0$

$\Rightarrow$ $L(x) \geq f(x)$ for $x > 0$

$\Rightarrow$ $L(998) \geq f(998)$ i.e. it is an $\boxed{\text{over estimate.}}$

Differential notation: $dy = f'(x)\,dx$

$$\boxed{dy = \frac{1}{3}x^{-\frac{2}{3}}dx}$$

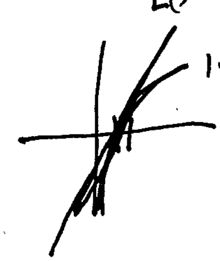
Q: Approximate $\ln(1.2)$.

Step 1: Find a point where you can evaluate the function:
close-by

Let $f(x) = \ln(x)$.

Note $f(1) = \ln(1) = 0$.

Step 2: Make a linear approx.



$$f'(x) = \frac{1}{x}$$

$$L(x) = f(1) + f'(1)(x-1) = 0 + 1(x-1) = x-1$$

Step 3: Plug in $x = 1.2$.

$$L(1.2) = 1.2 - 1 = \boxed{0.2}$$

$f''(x) = \frac{-1}{x^2} \Rightarrow \ln(x)$ is concave down

$\Rightarrow$ This is an over-estimate.

Q: Approx $\sin(3°)$.

Important first step: convert to radians

$$3° = 3 \cdot \frac{2\pi}{360} = \frac{\pi}{60} \text{ radians.}$$

Approx $\sin\left(\frac{\pi}{60}\right)$

Step 1: ~~Note~~ Let $f(x) = \sin(x)$, note $f(0) = \sin(0) = 0$.

Step 2: Linear approx:

$$f'(x) = \cos(x)$$

$$L(x) = f(0) + f'(0)(x-0)$$

$$= 0 + \cos(0) \cdot x$$

$$= x$$

Step 3: Plug in $x = \frac{\pi}{60}$

$$L\left(\frac{\pi}{60}\right) = \boxed{\frac{\pi}{60}}$$

Q: Approx the change in volume of a sphere when the radius changes from 11 meters to 13 meters.

Recall: $\Delta y \approx f'(a) \cdot \Delta x$

Here $y = f(x) =$ volume of sphere of radius x.

$= \frac{4}{3}\pi x^3$

$\Delta x = 13 - 11 = 2.$

$f'(x) = \frac{4}{3}\pi \cdot 3x^2 = 4\pi x^2$

$\Rightarrow \boxed{\Delta y \approx f'(11) \cdot 2 = 4\pi \cdot 11^2 \cdot 2}$