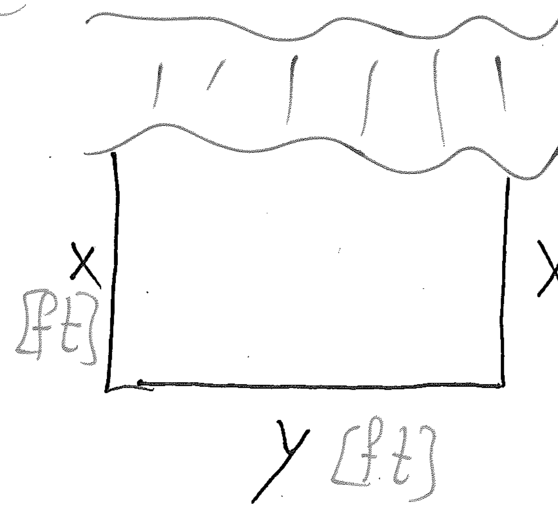
Guidelines for Optimization Problems

1. Read the problem carefully, identify the variables, and organize the given information with a picture.
2. Identify the objective function (the function to be optimized). Write it in terms of the variables of the problem.
3. Identify the constraint(s). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function.
5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.
6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.
7. Units.
8. Reflect.

Question 1.
A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He does not need a fence along the river. What are the dimensions of the field that has the largest area?

1



2 Target function:

$$A = X \cdot y$$

3 $$2X + y = 2{,}400 \text{ ft.}$$

4 $$y = 2400 - 2X$$

$$A(x) = X \cdot (2400 - 2x) = -2x^2 + 2400X$$

5 $$0 \le y \le 2400$$

$$0 \le X \le 1200$$

6 <u>opt I</u>: closed interval method:

$$A'(x) = -4X + 2400 \overset{!}{=} 0$$

CP: $\boxed{X = 600}$

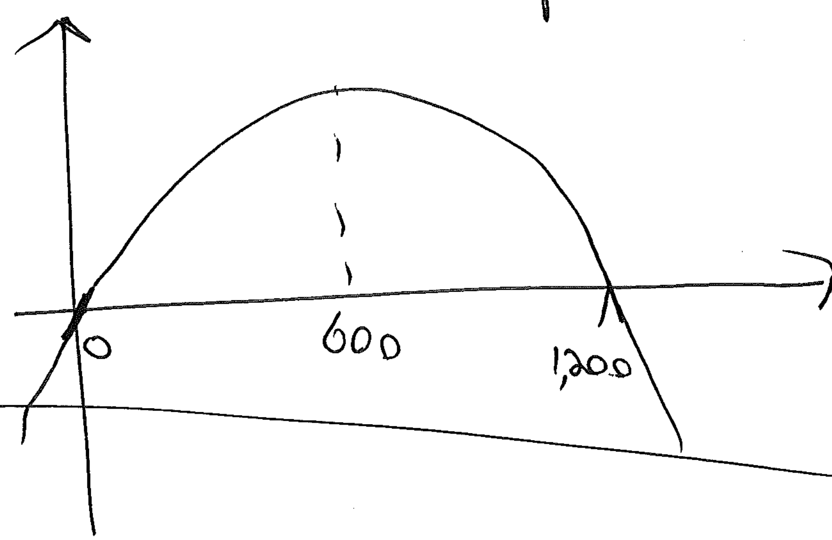
$$A(0) = 0 \qquad A(1200) = 0 \qquad A(600) = 720{,}000 > 0 \quad (ft^2)$$

The absolute maximum of A is at $\boxed{x = 600. \ [ft]}$

$$\boxed{y = 1{,}200 \ [ft]}$$

<u>Opt II:</u> $A(x) = x \cdot y = x(2400 - 2x) = -2x^2 + 2400x$

is a concave down parabula



<u>Opt III:</u> $A'(x) = 0$ only at $x = 600$.

$A(x)$ has only one loc. ext. so it is an absolute extremum.

Since $x = 600$ is a loc. max. it is a global max.

Question 2.
Find two numbers whose difference is 100 and whose product is a minimum.

x, y are the two numbers.

Target function: $P = x \cdot y$

Constraint: $\cancel{x-y}$ $y - x = 100$.

$$y = 100 + x$$

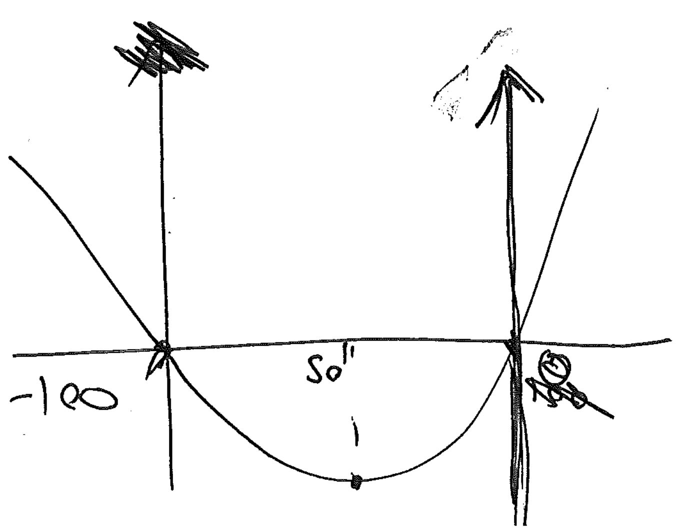
$$P(x) = x \cdot y = x(100 + x) = x^2 + 100x$$

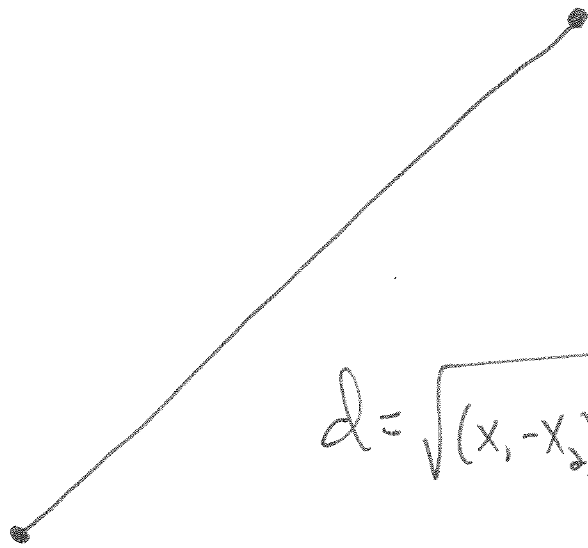
Concave up parabula.

$$\frac{dp}{dx} = 2x + 100 \overset{!}{=} 0$$

$$\boxed{x = -50}$$

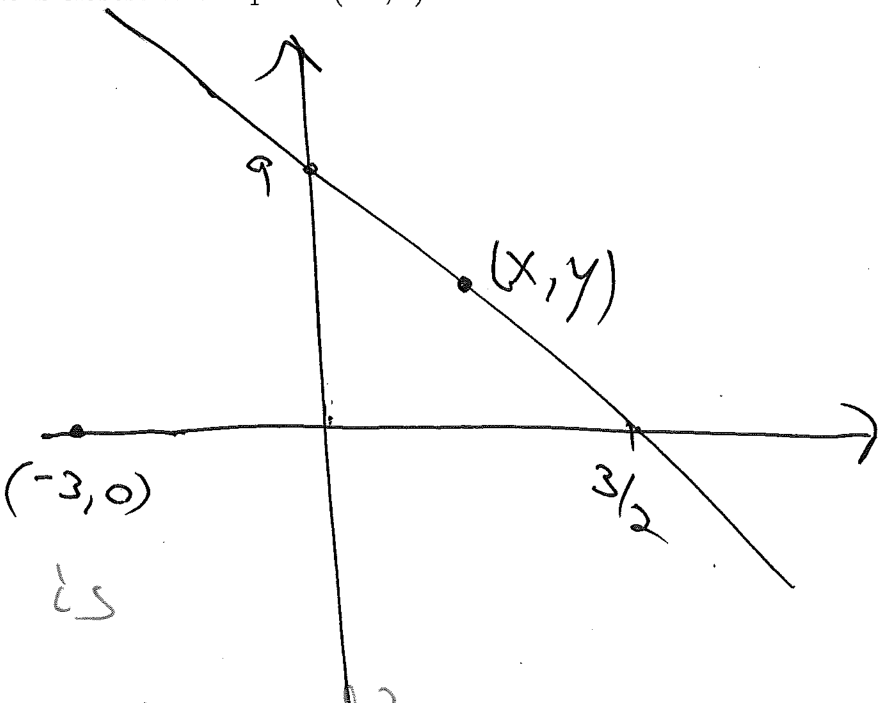
$$\boxed{y = 50}$$





$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Question 3.
Find the point of the line $6x + y = 9$ that is closest to the point $(-3, 0)$.



Target function:

$$d = \sqrt{(x+3)^2 + y^2}$$

$d > 0$

So minimizing d is the same as minimizing d^2.

$$T = (x+3)^2 + y^2$$

Constraints: $6x + y = 9$

$$y = 9 - 6x$$

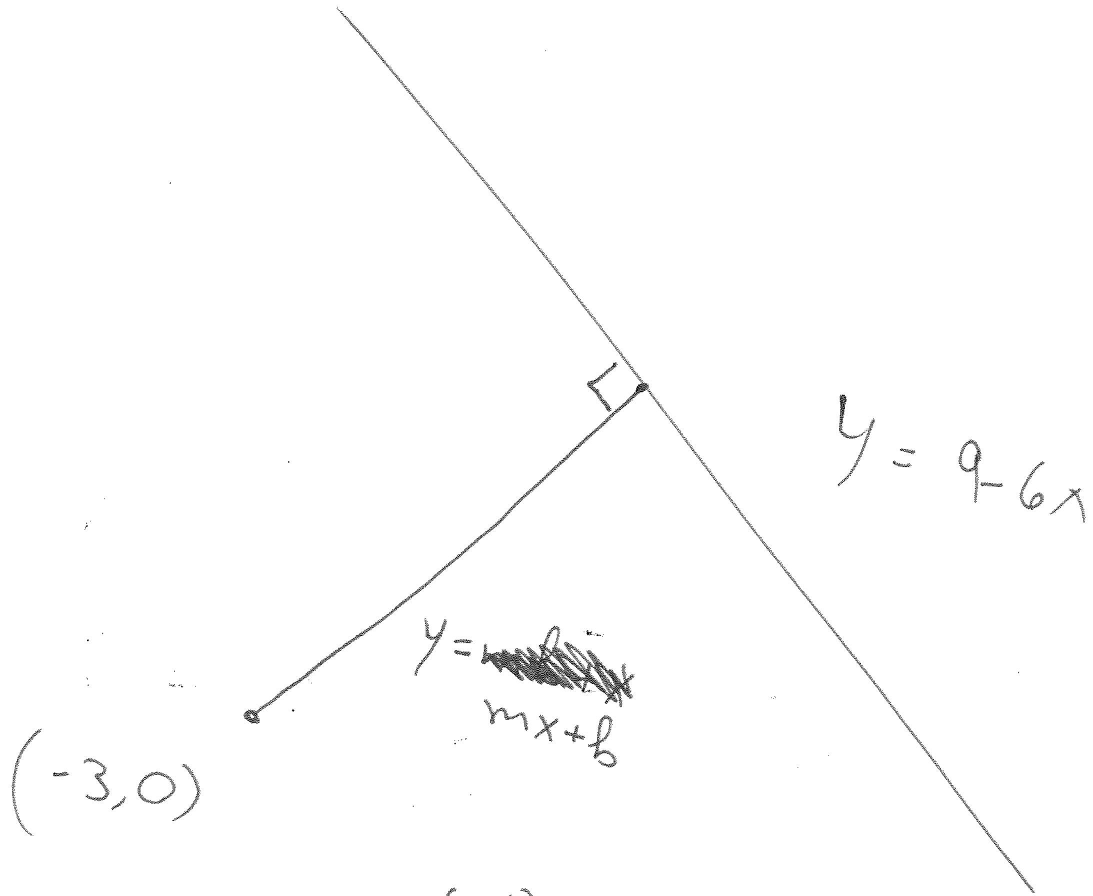
$$T(x) = (x+3)^2 + (9-6x)^2$$

$$T'(x) = 2(x+3) + 2(9-6x)(-6) = 74x - 102 \stackrel{!}{=} 0$$

$$x = \frac{102}{74} = \frac{51}{37}. \quad y = 9 - 6 \cdot \frac{51}{37}.$$

The only critical point.

T(x) a concave up So its a abs min.



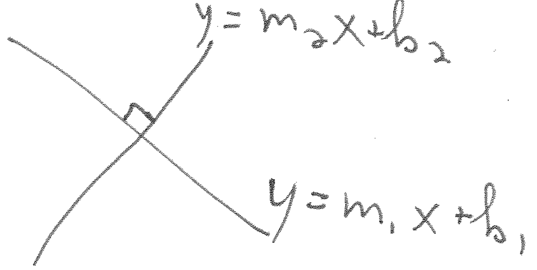
$$m \cdot (-6) = -1$$

$$m = \frac{1}{6}$$

$$y = \frac{1}{6}(x-(-3))+0$$

$$= \frac{1}{6}(x+3)$$

Recall



$$m_1 \cdot m_2 = -1$$

$$\begin{cases} 6x+y=9 \\ -\frac{1}{6}x+y=\frac{1}{2} \end{cases} \rightsquigarrow \text{The same solution.}$$

Question 4.
A cylindrical can is being made to contain 1 L of oil. Find the dimensions that will minimize the amount of metal needed to make the can.

$1\,L = 1{,}000\ cm^3$

Target function:

$$A = \underbrace{2\pi r h}_{\text{body}} + 2 \cdot \underbrace{\pi r^2}_{\text{top/bottom}}$$

Constraints:

$$V = \pi r^2 h = 1{,}000 \quad [cm^3]$$

$$h = \frac{1000}{\pi r^2}$$

$$\boxed{0 < r}$$

$$A(r) = 2\cancel{\pi}\cancel{r} \cdot \frac{1000}{\cancel{\pi} r^{\cancel{2}}} + 2\pi r^2 = \frac{2000}{r} + 2\pi r^2$$

$$\frac{dA}{dr} = -\frac{2000}{r^2} + 4\pi r \overset{!}{=} 0 \qquad r^3 = \frac{500}{\pi}$$

$$\boxed{r = \sqrt[3]{\frac{500}{\pi}}\ [cm]} \quad \boxed{h = \frac{1000}{\pi \sqrt[3]{\frac{500}{\pi}}^{\,2}}\ [cm]}$$

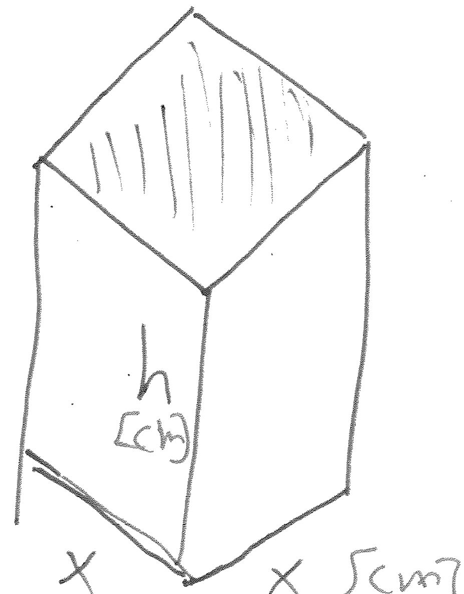
Why is this an abs. min?

Question 5.
If 1200 cm^2 of material is available to make a box with a square base and open top, find the largest possible volume of the box.

<u>Terget function</u>:

$V = x^2 h$

<u>Constraints</u>:



$A = x^2 + 4 \cdot x h = 1200\ [cm^2]$

$h = \dfrac{1200 - x^2}{4x}$

$0 < x$

$V(x) = x^2 \cdot \dfrac{1200 - x^2}{4\cancel{x}} = \frac{1}{4} \cdot x \cdot (1200 - x^2) = \frac{1}{4}(1200x - x^3)$

$\dfrac{dV}{dx} = \frac{1}{4}(1200 - 3x^2) \overset{!}{=} 0$

$x^2 = 400$

$x = \pm 20$

Only $x = +20$ is in the domain.

$x = 20\ [cm]$

$h = \ldots$

$V(20) = \ldots\ [cm^3]$.

$X = 20$ is not the only loc. ext.

of $V(x)$ on $(-\infty, \infty)$.

But it is the only loc. ext. of $V(x)$ on $(0, \infty)$ so it is ~~the~~ an absolute ext. on $(0, \infty)$

It is a local max hence an absolute max. for $(0, \infty)$.