Curve Sketching (4.3)

Goal: Given $f(x)$, sketch $y = f(x)$ (the graph)

Strategy:

1) Find domain
2) Find asymptotes (vertical/horizontal)
3) Crit pts, $(x, f(x))$, & find sign(f').
4) Inflection pts, sign f''.
5) Tabulate above information.
6) Sketch the graph.

Ex 1) Sketch $y = f(x) = \dfrac{2x^2}{x^2-1}$

Step 1: $x \neq \pm 1$ Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Step 2: Vertical asymptotes at $x = \pm 1$

Horizontal: $\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \pm\infty} \dfrac{2}{1 - \frac{1}{x^2}} = 2 = y$

Step 3: $f'(x) = \dfrac{(x^2-1) \cdot 4x - 2x^2 \cdot 2x}{(x^2-1)^2} = \dfrac{4x^3 - 4x - 4x^3}{(x^2-1)^2}$

$= \dfrac{-4x}{(x^2-1)^2}$

Crit pt: $x=0$, $\text{sign}(f') = \dfrac{\text{sign}(-4)\cdot\text{sign}(x)\cdot}{\text{sign}\left((x^2-1)^2\right)}$

$$= \frac{(-)\cdot\text{sign}(x)}{+} = -\text{sign}(x)$$

$f'(x)$ is $\begin{cases} >0 & \text{for } x<0 \\ <0 & \text{for } x>0 \end{cases}$

Step 4: $f''(x) = \dfrac{d}{dx} f'(x) = \dfrac{(x^2-1)^2(-4) - (-4x)\,2(x^2-1)\cdot 2x}{(x^2-1)^4}$

$$= \ldots = \frac{12x^2+4}{(x^2-1)^3}$$

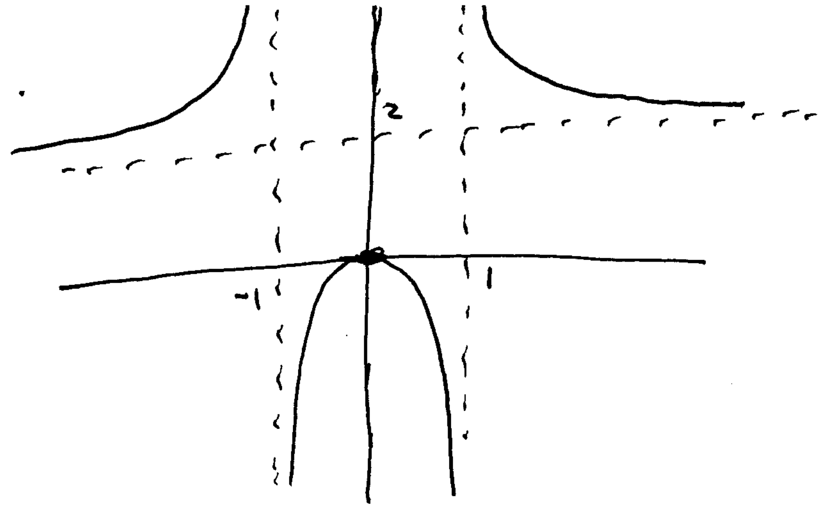
$$\text{sign}(f'') = \frac{\text{sign}(12x^2+4)}{\text{sign}\left((x^2-1)^3\right)} = \text{sign}(x^2-1)$$

$$= \begin{cases} + & x>1 \\ - & -1<x<1 \\ + & x<-1 \end{cases}$$

Step 5:

	$-\infty$		-1		0		1		∞
x		$(-\infty,-1)$		$(-1,0)$		$(0,1)$		$(1,\infty)$	
f'		$+$		$+$		$-$		$-$	
f''		$+$		$-$		$-$		$+$	
f									
	$\boxed{f\to 2}$		$\boxed{\text{vert}}$		$\boxed{f'=0,\ f(0)=0}$		$\boxed{\text{vert}}$		$\boxed{f\to 2}$

Step 6: Graph.



Ex 2) Sketch $y = f(x) = \dfrac{e^x}{x^2}$

Step 1: $x \neq 0$ Domain: $(-\infty, 0) \cup (0, \infty)$

Step 2: Vertical: $x = 0$

Horizontal: $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{e^x}{x^2} = \infty$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{e^x}{x^2} = 0$

Step 3: $f'(x) = \dfrac{x^2 \cdot e^x - e^x \cdot 2x}{x^4} = \dfrac{(x-2)e^x}{x^3}$

crit pt: $x = 2$

$\text{sign}(f'(x)) = \text{sign}(x-2) \cdot \text{sign}(x)$

x	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
f'	+	−	+

Step 4: $f''(x) = \dfrac{d}{dx} f(x) = \dots\dots = \dfrac{e^x}{x^4}(x^2 - 4x + 6)$

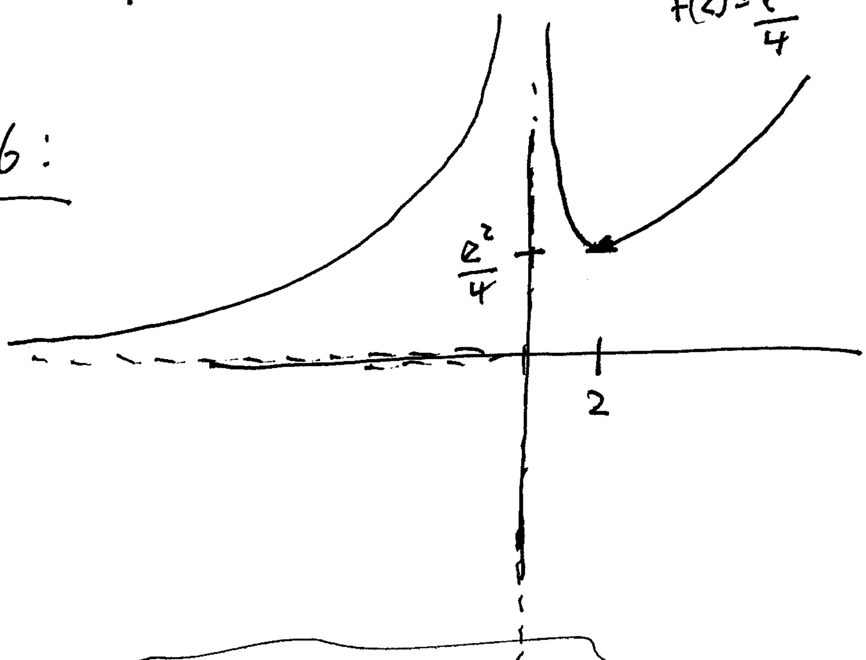
Exercize for you: Show that
$x^2 - 4x + 6 > 0$ for all x.
$\Rightarrow f'' > 0$ for all x in domain.

Step 5:

x	$-\infty$	$(-\infty, 0)$	0	$(0, 2)$	2	$(2, \infty)$	∞
f'		$+$		$-$		$+$	
f''		$+$		$+$		$+$	
f	$\boxed{f \to 0}$		$\boxed{\text{vert}}$		$\boxed{f' = 0,\ f(2) = e^2/4}$		$\boxed{f \to \infty}$

$f(2) = \frac{e^2}{4}$

Step 6:



Exercize for you?
Let $g(x) = x^2 - 4x + 6$
Note: $g'(x) = 2x - 4 \Rightarrow$ crit pt at $x = 2$
$g''(x) = 2 \Rightarrow$ By 2nd deriv test that
$(2, g(2))$ is a local min. Only $1 \Rightarrow$
Absolute min. Also $g(2) = 4 - 8 + 6 = 2 \Rightarrow g(x) \geq 2$ for all x.

Ex) Sketch $y = f(x) = \frac{x}{\sqrt{x^2-1}}$

step 1: $x \neq \pm 1$, $x^2 - 1 \geq 0 \Rightarrow x \geq 1$ or $x \leq -1$

Domain: $(-\infty, -1) \cup (1, \infty)$

Step 2: Vertical: $x = \pm 1$

Horizontal:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2(1-\frac{1}{x^2})}} = \lim_{x\to\infty} \frac{x}{|x|\sqrt{1-\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1$$

Similarly, $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{|x|\sqrt{1-\frac{1}{x^2}}} = -1$

Step 3: Note: $f(x) = x(x^2-1)^{-1/2}$

$$f'(x) = (x^2-1)^{-1/2} + x\left(-\tfrac{1}{2}\right)(x^2-1)^{-3/2} \cdot 2x$$

$$= \frac{1}{(x^2-1)^{1/2}} + \frac{-x^2}{(x^2-1)^{3/2}} = \frac{x^2-1}{(x^2-1)^{1/2}(x^2-1)} + \frac{-x^2}{(x^2-1)^{3/2}}$$

$$= \frac{x^2-1-x^2}{(x^2-1)^{3/2}} = \frac{-1}{(x^2-1)^{3/2}}$$

No crit pts

sign$(f') = -$ where f' is defined

Step 4: Note: $f'(x) = -(x^2-1)^{-3/2}$

$$\Rightarrow f''(x) = -\left(-\tfrac{3}{2}\right)\cdot(x^2-1)^{-5/2} \cdot 2x = 3x(x^2-1)^{-5/2}$$

$$\operatorname{sign}(f'') = \operatorname{sign}(x) = \begin{cases} - & x < 0 \\ + & x > 0 \end{cases}$$

Step 5:

x	$-\infty$	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,\infty)$	∞
f'		$-$		DNE		$-$	
f''		$-$		DNE		$+$	
f	$\boxed{f\to -1}$	⌢	$\boxed{\text{vert}}$	DNE	$\boxed{\text{vert}}$	⌣	$\boxed{f\to 1}$

Step 6: