Exponential growth (See "Section 6.8 for week 6" from common course webpage.)

Recall: $\frac{d}{dt} e^t = e^t$, $\frac{d}{dt} e^{kt} = k e^{kt}$, $\frac{d}{dt}\left(C e^{kt}\right) = C \cdot \frac{d}{dt}\left(e^{kt}\right) = C \cdot k \cdot e^{kt}$

Exponential growth function:

$$y(t) = y_0 \cdot e^{kt} \qquad t \geq 0, \; k \geq 0$$

y_0: value of $y(0)$ (a constant)

k: "rate constant"

, Ex) $y(t) = 10 e^{0.1 \cdot t}$

Observe: $\frac{dy}{dt} = \frac{d}{dt}\left(y_0 e^{kt}\right) = y_0 \cdot k e^{kt} = k \cdot y$

I.e., the rate of change is proportional to the function value.

Def.

$$\boxed{\begin{array}{l} \text{Growth rate} = \frac{dy}{dt} \\ \text{Relative growth rate} = \frac{1}{y}\frac{dy}{dt} \end{array}}$$

For exp. growth func.

$$\frac{dy}{dt} = k \cdot y$$

$$\frac{1}{y}\frac{dy}{dt} = k$$

I.e., the relative growth rate is constant!
(for exp. growth functions.)

Ex) Suppose the population of bunnies on an island is $b(t) = 5 \cdot e^{0.02t}$ and the population of people is $p(t) = 100 + 35t$, $t \geq 0$ is measured in years.

Growth rates? Relative growth rates?

<u>Growth rates:</u>

- $\frac{db}{dt} = y_0 k e^{kt} = 5 \cdot 0.02 \cdot e^{0.02t} = 0.1 \cdot e^{0.02t}$
- $\frac{dp}{dt} = \frac{d}{dt}(100 + 35t) = 35 \leftarrow$ const.

<u>Relative growth rates:</u>

- $\frac{1}{b} \cdot \frac{db}{dt} = k = 0.02 \leftarrow$ const.
- $\frac{1}{p} \cdot \frac{dp}{dt} = \frac{35}{100 + 35t}$

Q: • Population A increases at a rate of 1% per year.

• Population B increases by 1 million people per year.

Which exhibits exponential growth? $\boxed{\text{Pop. A}}$

Q: How long does it take for an exponential growth function to double?

Find t_0, t_1 so that $y(t_1) = 2y(t_0)$.

Then doubling time is $t_1 - t_0$.

Plug in exp. growth funct. to give:

$$y(t_1) = 2y(t_0)$$

$$\Rightarrow \cancel{y_0} e^{kt_1} = 2 \cdot \cancel{y_0} e^{kt_0}$$

Take ln of both sides: $\ln(e^{kt_1}) = \ln(2e^{kt_0})$

$$\Rightarrow kt_1 = \ln(2) + \ln(e^{kt_0}) = \ln(2) + kt_0$$

$$\Rightarrow k(t_1 - t_0) = \ln(2)$$

$$\Rightarrow \boxed{t_1 - t_0 = \frac{\ln(2)}{k}}$$ ← Doubling time for exp. growth function.

Ex) 100 bunnies are introduced to an island and, from there, the population grows exponentially. After 5 years, there are 300 bunnies.

a) What is the population after t years?

(I.e., find y_0 & k in model $y(t) = y_0 \cdot e^{kt}$.)

We have 2 unknowns, we need 2 data points:

$y(0) = 100$

$\Rightarrow y_0 \cdot \cancel{e^{k \cdot 0}} = 100$

$\Rightarrow \boxed{y_0 = 100}$

$y(5) = 300$

$\Rightarrow y_0 \cdot e^{k5} = 300$

$\Rightarrow \cancel{100} e^{k \cdot 5} = 3\cancel{00}$

$\Rightarrow e^{k \cdot 5} = 3$

$\Rightarrow k \cdot 5 = \ln 3$

$\boxed{k = \frac{\ln 3}{5}}$

Thus, population after t years is

$\boxed{y(t) = 100\, e^{\frac{\ln 3}{5} t}}$

Note: $\frac{\ln 3}{5} \approx 0.22$, so $\boxed{y(t) \approx 100\, e^{0.22t}}$

b) What is the doubling time?

$$\frac{\ln 2}{k} = \frac{\ln 2}{\left(\frac{\ln 3}{5}\right)} = \boxed{\frac{5\ln 2}{\ln 3}}$$

c) What will be the population after 15 years?

$$y(15) = 100 \cdot e^{(0.22 \cdot 15)} = 2711 \longleftarrow \text{rounding } k$$

or

$$y(15) = 100 \cdot e^{\left(\frac{\ln 3}{5} \cdot 15\right)} = 100 \cdot e^{(\ln 3) \cdot 3} = 100\, e^{\ln(3^3)} = 100 \cdot 3^3$$

$$= 100 \cdot 27 = \boxed{2700} \longleftarrow \text{exact expression for } k$$

Do this

d) What is the growth rate at 10 years?

$$y'(10) = k \cdot y(t)\Big|_{t=10} = \frac{\ln 3}{5} \cdot 100 \cdot e^{\frac{\ln 3}{5} \cdot 10} = \ln(3) \cdot 20 \cdot e^{(\ln 3) \cdot 2}$$

$$= \ln(3) \cdot 20 \cdot e^{\ln(3^2)} = \ln(3) \cdot 20 \cdot 3^2 = \boxed{\ln(3) \cdot 180}$$

Continuously Compounded interest (Notes on common course webpage)

Recall: (Compounded interest formula)

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

P: principal
r: annual interest rate (as a fraction)
n: # of compounding periods per year
t: years
A: Value after t years

Simple case: $n=1$, $r=0.5$

$$A = P \cdot 1.5^{t}$$

or $n=2$, $r=0.5$.

$$A = P \cdot (1.25)^{2t}$$

Ex) Invest \$2000 at 7% annual interest compounded semi-annually (twice a year). Then, after 8 years we have

$$\boxed{A = 2000\left(1+\frac{0.07}{2}\right)^{2\cdot 8} \text{ dollars}}$$

What happens when you increase the number of compounding periods, n?

Consider special case $r=1$.

<u>If $n=1$,</u>

$A = P\,2^t$ (doubles each year)

<u>If $n=2$,</u>

$A = P\left(1+\frac{1}{2}\right)^{2t} = P\left((1.5)^2\right)^t = P\,2.25^t$ (multiplies by 2.25 each year.)

<u>If $n \to \infty$,</u>

$$A = \lim_{n\to\infty} P\left(1+\frac{1}{n}\right)^{nt} = P\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)^t = Pe^t$$

(multiplies by $e = 2.71\ldots$ each year.)

$\boxed{\underline{\text{Fact:}}\ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$

Generally, when we take the limit as $n \to \infty$, we have $\boxed{A = Pe^{rt}}$

We call this <u>continuously compounded interest.</u>

<u>Rmk</u> This is an example of exponential growth.

Ex) Suppose you deposit \$500 in a savings account that has an APY of 6.8% per year with continuous compounding. How long does it take to reach a balance of \$2,500?

Def APY = "Annual percentage yield" = percentage increase in balance over the course of a year.

Note: $APY \neq \frac{r}{100}$ unless $n=1$

Step 1: Calculate constants P & r defining growth function $A = Pe^{rt}$.

$A(0) = P = 500$

$\Rightarrow A(t) = 500e^{rt}$

$APY = 6.18\%$

$A(1) = (1.0618)A(0)$: in words, "we gain 6.18% over 1st yr"

$\Rightarrow \cancel{500}e^{r \cdot 1} = 1.0618 \cdot \cancel{500}$

$\Rightarrow r = \ln(1.0618) \approx 0.060$

$\Rightarrow \boxed{A(t) \approx 500e^{0.060 \cdot t}}$

How long to reach \$2500?

Solve $A(t) = 2500$

$\Rightarrow \cancel{500} e^{0.060t} = \cancel{2500}\ 5$

$\Rightarrow 0.060t = \ln 5$

$\Rightarrow \boxed{t = \frac{\ln 5}{0.060} \approx 26.8 \text{ yrs}}$