MATH 223 Assignment #1

1. Let $A = \begin{bmatrix} 0 & x \\ 1 & y \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$. Determine all x, y so that $AB = BA$.

2. Find a 2×2 matrix A, no entry of which is 0, with $A^2 = A$. Note that your first guesses $A = I$ or $A = 0$ (or indeed $A = E_{11}$) have 0 entries.

3. Darryl has a sum of s dollars in t coins. The coins are either \$.25 or \$.10 Express the number x of quarters (\$.25 coins) and the number y of dimes (\$.1 coins) in terms of s and t.

4. Assume you are given a pair of matrices A, B which satisfy $AB = BA$. Show that if we set $C = A^2 + 2A$ and $D = B^3 + 5I$, then $CD = DC$. Then try to generalize this in some interesting way, namely find a property so that for matrices C, D with that certain property, then $CD = DC$. For example $C = A^2 + 6A$ and $D = 3B^3 - 2I$ will also have $CD = DC$.

5. Let $R(\theta)$ denote the matrix of the transformation which rotates the plane by θ counterclockwise around the origin. Explain in words (using transformations) why $R(\theta)R(\phi) = R(\theta + \phi)$. Show how you can use this to derive the formulas for $\cos(\theta + \phi), \sin(\theta + \phi)$ in terms of $\cos(\theta), \sin(\theta), \cos(\phi), \sin(\phi)$.

6. Find the matrix A associated with the linear transformation T that has $T\binom{1}{1} = \binom{2}{3}$ and $T\binom{2}{3} = \binom{5}{4}$.

7. a) Assume A, B are 2×2 invertible matrices so that A^{-1} and B^{-1} exist.
Show that $(AB)^{-1} = B^{-1}A^{-1}$.
b) Given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then define } A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix},$$

where A^T is called the *transpose* of A. The dot product of two vectors $\mathbf{x} = \binom{a}{b}, \mathbf{y} = \binom{c}{d}$ is $\mathbf{x} \cdot \mathbf{y} = ac + bd$. Then the i, j entry of AB is the dot product of the ith row of A and the jth column of B. Using this idea, show that $(AB)^T = B^T A^T$. (One could verify $(AB)^T = B^T A^T$ for two arbitrary 2×2 matrices A, B directly but the argument wouldn't generalize to larger matrices).

8. Consider two nonzero vectors $\mathbf{x} = \binom{a}{b}, \mathbf{y} = \binom{c}{d}$. Then there is a θ with $0 \leq \theta < 2\pi$ and a $\rho > 0$ so that $\mathbf{y} = \rho R(\theta)\mathbf{x}$. Use our knowledge of rotation matrices to establish a simple condition on a, b, c, d so that the angle θ satisfies $0 < \theta < \pi$. You may assume a, b, c, d are nonzero, if that assists you, and even assume the two vectors $\mathbf{x}, \mathbf{y}$ have the same length ($\rho = 1$).