

Math 100 – SOLUTIONS TO WORKSHEET 3
LIMITS AT INFINITY; CONTINUITY

1. THE SQUEEZE THEOREM

(1) $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right)$.

Solution: Since $-1 \leq \sin \theta \leq 1$ for all θ while $x^2 \geq 0$ we have for all x that

$$-x^2 \leq x^2 \sin\left(\frac{\pi}{x}\right) \leq x^2.$$

Now $\lim_{x \rightarrow 0} x^2 = 0$ and $\lim_{x \rightarrow 0} (-x^2) = 0$, so by the sandwich theorem $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0$ too.

(2) (Final, 2014) Suppose that $8x \leq f(x) \leq x^2 + 16$ for all $x \geq 0$. Find $\lim_{x \rightarrow 4} f(x)$.

Solution: We have $\lim_{x \rightarrow 4} 8x = 32$ and $\lim_{x \rightarrow 4} x^2 + 16 = 32$ so by the sandwich theorem $\lim_{x \rightarrow 4} f(x)$ exists and equals 32.

2. LIMITS AT INFINITY

(1) Evaluate the following limits:

(a) $\lim_{x \rightarrow \infty} \frac{x^2+1}{x-3} =$

Solution: $\lim_{x \rightarrow \infty} \frac{x^2+1}{x-3} = \lim_{x \rightarrow \infty} \frac{x^2}{x} \cdot \frac{1+\frac{1}{x^2}}{1-\frac{3}{x}} = \lim_{x \rightarrow \infty} x \cdot \frac{1+\frac{1}{x^2}}{1-\frac{3}{x}} = \infty$.

(b) (Final, 2015) $\lim_{x \rightarrow \infty} \frac{x+1}{x^2+2x-8} =$

Solution: $\lim_{x \rightarrow \infty} \frac{x+1}{x^2+2x-8} = \lim_{x \rightarrow \infty} \frac{x}{x^2} \cdot \frac{1+\frac{1}{x}}{1+\frac{2}{x}-\frac{8}{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{x} \cdot \frac{1+\frac{1}{x}}{1+\frac{2}{x}-\frac{8}{x^2}} = 0$.

(c) (Quiz, 2015) $\lim_{x \rightarrow -\infty} \frac{3x}{\sqrt{4x^2+x-2x}} =$

Solution: We have

$$\begin{aligned} \frac{3x}{\sqrt{4x^2+x-2x}} &= \frac{3x}{\sqrt{x^2\left(4+\frac{x}{x^2}\right)-2x}} = \frac{3x}{\sqrt{x^2}\sqrt{4+\frac{1}{x}-2x}} \\ &= \frac{3x}{|x|\sqrt{4+\frac{1}{x}-2x}} = \frac{3x}{(-x)\sqrt{4+\frac{1}{x}-2x}} \\ &= \frac{3}{-\sqrt{4+\frac{1}{x}-2}} \xrightarrow{x \rightarrow -\infty} \frac{3}{-\sqrt{4+0-2}} = \boxed{-\frac{3}{4}}. \end{aligned}$$

(d) $\lim_{x \rightarrow \infty} \frac{\sqrt{x^4+\sin x}}{x^2-\cos x} =$

Solution: $\lim_{x \rightarrow \infty} \frac{\sqrt{x^4+\sin x}}{x^2-\cos x} = \lim_{x \rightarrow \infty} \frac{x^2\sqrt{1+\frac{\sin x}{x^4}}}{x^2\left(1-\frac{\cos x}{x^2}\right)} = \lim_{x \rightarrow \infty} \frac{\sqrt{1+\frac{\sin x}{x^4}}}{1-\frac{\cos x}{x^2}}$. Now for all x we have $-\frac{1}{x^4} \leq \frac{\sin x}{x^4} \leq \frac{1}{x^4}$ and $-\frac{1}{x^2} \leq \frac{\cos x}{x^2} \leq \frac{1}{x^2}$. Since $\lim_{x \rightarrow \infty} \frac{1}{x^2} = \lim_{x \rightarrow \infty} \frac{1}{x^4} = 0$ by the squeeze theorem we have $\lim_{x \rightarrow \infty} \frac{\sin x}{x^4} = \lim_{x \rightarrow \infty} \frac{\cos x}{x^2} = 0$. Thus

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^4+\sin x}}{x^2-\cos x} = \frac{\sqrt{1+0}}{1-0} = \boxed{1}.$$

(e) $\lim_{x \rightarrow -\infty} (\sqrt{x^2+2x} - \sqrt{x^2-1}) =$

Solution: We have

$$\begin{aligned} \sqrt{x^2 + 2x} - \sqrt{x^2 - 1} &= \sqrt{x^2 + 2x} - \sqrt{x^2 - 1} \cdot \frac{\sqrt{x^2 + 2x} + \sqrt{x^2 - 1}}{\sqrt{x^2 + 2x} + \sqrt{x^2 - 1}} = \frac{(x^2 + 2x) - (x^2 - 1)}{\sqrt{x^2 + 2x} + \sqrt{x^2 - 1}} \\ &= \frac{2x + 1}{|x| \sqrt{1 + \frac{2}{x}} + |x| \sqrt{1 - \frac{1}{x^2}}} = \frac{x(2 + \frac{1}{x})}{-x \sqrt{1 + \frac{2}{x}} + (-x) \sqrt{1 - \frac{1}{x^2}}} \\ &= -\frac{2 + \frac{1}{x}}{\sqrt{1 + \frac{2}{x}} + \sqrt{1 - \frac{1}{x^2}}} \xrightarrow{x \rightarrow -\infty} \frac{2}{1 + 1} = \boxed{1}. \end{aligned}$$

3. CONTINUITY

(1) Which of these functions are continuous everywhere? Why?

(a) $f(x) = \begin{cases} x & x < 0 \\ \cos x & x \geq 0 \end{cases}$

Solution: This is clearly continuous on $(-\infty, 0)$ and $(0, \infty)$ but $\lim_{x \rightarrow 0^-} f(x) = 0$ while $\lim_{x \rightarrow 0^+} f(x) = 1$.

(b) $f(x) = \begin{cases} x & x < 0 \\ \sin x & x \geq 0 \end{cases}$

Solution: This is clearly continuous on $(-\infty, 0)$ and $(0, \infty)$. Also, $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = 0 = f(0)$ so the function is continuous everywhere.

(2) Let $f(x) = \frac{x^3 - x^2}{x - 1}$.

(a) Why is $f(x)$ discontinuous at $x = 1$?

Solution: The formula is undefined there.

(b) Find b such that $g(x) = \begin{cases} f(x) & x \neq 1 \\ b & x = 1 \end{cases}$ is continuous everywhere.

Solution: $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^2(x-1)}{x-1} = \lim_{x \rightarrow 1} x^2 = 1$ so setting $b = 1$ gives the desired function.

(c) Find c, d such that $f(x) = \begin{cases} \sqrt{x} & 0 \leq x < 1 \\ c & x = 1 \\ d - x^2 & x > 1 \end{cases}$ is continuous.

Solution: The function is already continuous on $(-\infty, 1)$ and $(1, \infty)$. We have $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} \sqrt{x} = 1$ so we must have $c = 1$. We also need $1 = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (d - x^2) = d - 1$ so we need $d = 2$.

(d) (Final 2013) For which value of the constant c is $f(x) = \begin{cases} cx^2 + 3 & x \geq 1 \\ 2x^3 - c & x < 1 \end{cases}$ continuous on $(-\infty, \infty)$?

Solution: The function is already continuous on $(-\infty, 1)$ and $(1, \infty)$. We have $\lim_{x \rightarrow 1^-} f(x) = 2 \cdot 1^3 - c = 2 - c$ and $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} c \cdot 1^2 + 3 = 3 + c = f(1)$ so the function will be continuous at $x = 1$ iff $2 - c = 3 + c$ that is if $c = \frac{5}{2}$.

(3) Where are the following functions continuous?

(a) $\frac{1}{\sqrt{7-x^2}}$

Solution: $(-\sqrt{7}, \sqrt{7})$.

(b) $\frac{x^2 + 2x + 1}{2 + \cos x}$

Solution: $(-\infty, \infty)$

(c) $\frac{2 + \cos x}{x^2 + 2x + 1}$

Solution: The denominator vanishes at $x = 1$ where the function blows up ($2 + \cos 1 \neq 0$) so the domain is $(-\infty, -1) \cup (-1, \infty)$.

(d) $\log(\sin x)$

Solution: The logarithm is defined only for positive arguments, so the domain is $\{x \mid \sin x > 0\} = \bigcup_{k \in \mathbb{Z}} (2\pi k - \frac{\pi}{2}, 2\pi k + \frac{\pi}{2})$

- (4) (Final 2011) Suppose f, g are continuous such that $g(3) = 2$ and $\lim_{x \rightarrow 3} (xf(x) + g(x)) = 1$. Find $f(3)$.

Solution: Since f, g are continuous and applying the limit laws we have

$$\begin{aligned} 1 &= \lim_{x \rightarrow 3} (xf(x) + g(x)) = \left(\lim_{x \rightarrow 3} x\right) \left(\lim_{x \rightarrow 3} f(x)\right) + \left(\lim_{x \rightarrow 3} g(x)\right) \\ &= 3f(3) + g(3) = 3f(3) + 6. \end{aligned}$$

Solving for $f(3)$ we get

$$\boxed{f(3) = -\frac{5}{3}}.$$