Introduction to compressed sensing

Yaniv Plan

September 8, 2015

- Seminal papers: (Candès, Romberg, Tao 2004), (Donoho 2004)
- For the past 9 years: $\sim$2 new papers per day

Motivating Problem: Magnetic Resonance Imaging (MRI)

"If Bryce took a single breath, the image would be blurred. That meant deepening the anesthesia enough to stop respiration. It would take a full two minutes for a standard MRI to capture the image, but if the anesthesiologists shut down Bryce's breathing for that long, his glitchy liver would be the least of his problems." [Wired Magazine 2010]

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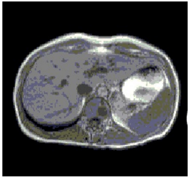

Figure: Standard MRI

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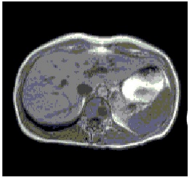
 $+$  $=$ 

Figure: Goal: reduce acquisition time

MRI: Singing protons!

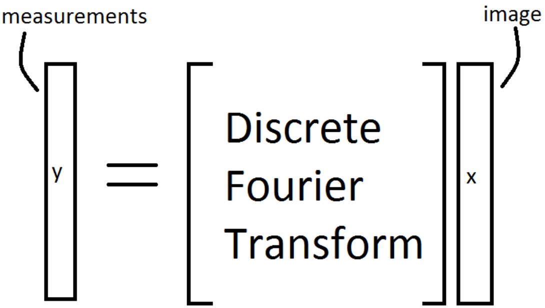


Figure: Standard MRI: Fully sampled DFT

- Image reconstruction: Invert the DFT

MRI: Singing protons!

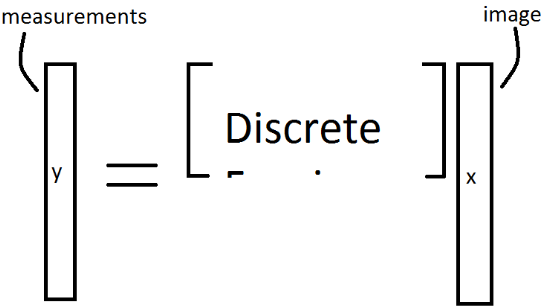


Figure: Goal: reconstruct image from part of the DFT

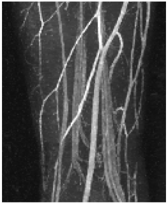
- Image reconstruction: Impossible?
- Is there any information that we have not used?
- Do images have structure?

Sparsity!

Sparse images

(a) Starry sky

(b) U. Mich.

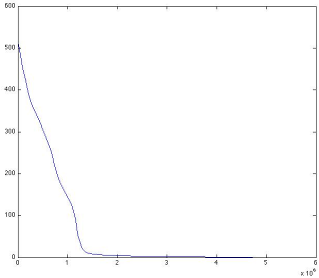


(c) Angiography (imaging blood veins)

- Are images generally sparse?

Signal sparsity

(a) Golden gate bridge



(b) Ordered wavelet coefficients

Images tend to be compressible, i.e. sparse in some basis.

- "Natural" images may be sparsified via the appropriate transform (wavelet, curvelet, shearlet...).
- Sparsity lives in audio signals, radar, statistical models, PDE solutions and much more.

Mathematics of sparsity

(Combining sparsity with undersampling)

Sparsity riddle

- Riddle: A sleeping dragon guards n locked treasure chests. One is filled with gold and the other $n-1$ are empty. You sneak into the dragon's lair carrying with you a large weighing machine. You wish to discover which chest holds the gold. How many times do you have to use the weighing machine?
- What if s of the chests contain gold, with $s \ll n$?

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  Ans: $\log n$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 100\text{ kg} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 100\text{ kg} \\ 100\text{ kg} \\ 0\text{ kg} \end{pmatrix}$$

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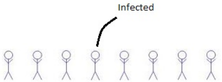
Group testing:



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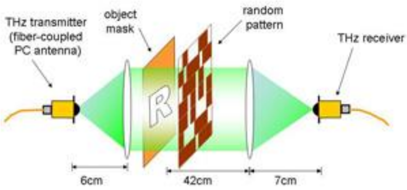
Single pixel camera (first made at rice, now sold at InView)



Mathematical formalism

Measurements:

$$y_i = \langle a_i, x \rangle + z_i =$$

where z is a noise term, and x is sparse.

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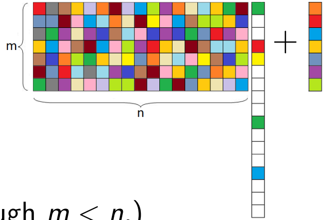
Matrix notation:

$$y = Ax + z =$$

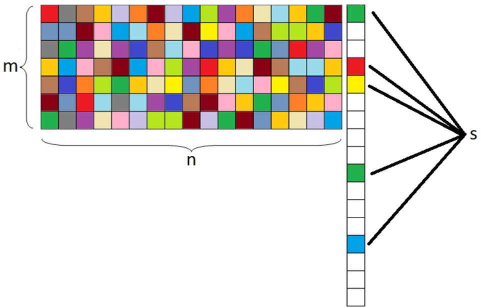


Goal: Estimate x. (Even though $m < n$.)

Noiseless lower bound

Question for the audience: What is a lower bound on the number of measurements m (rows of A) needed to recover x?

A $\qquad$ x



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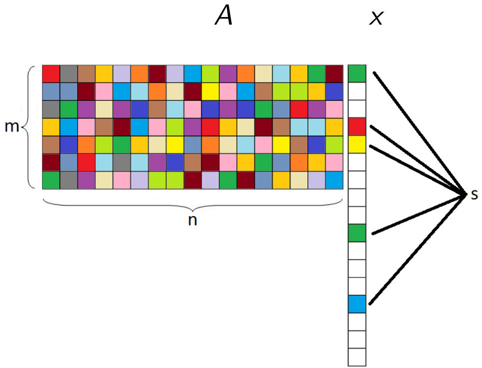

Answer: Recovery is impossible if there exists $x' \neq x$ such that $Ax = Ax'$ and $\|x'\|_0 \leq s$.
$\Rightarrow$ We need for $m \geq 2s$ for *uniform* recovery.

Can we reconstruct x with $O(s)$ measurements?

Can we reconstruct $\boldsymbol{x}$ with $O(s)$ measurements?

Reconstruction method:

$$\min \left\|\boldsymbol{x}'\right\|_0 \quad \text{such that} \quad \boldsymbol{A}\boldsymbol{x}' = \boldsymbol{y} \tag{0.1}$$

- $\|\boldsymbol{x}'\|_0 :=$ number of non-zero entries of $\boldsymbol{x}'$.

What are conditions on $\boldsymbol{A}$ that allow reconstruction of $\boldsymbol{x}$?

Bad: Let A_i be the i-th column of $\boldsymbol{A}$ and suppose $A_1 = A_2$. Then $Ae_1 = Ae_2 \Rightarrow$ reconstruction of sparse vectors is impossible.

Incoherence condition

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Definition (Incoherence)

Suppose (w.l.o.g.) that $\boldsymbol{A}$ is column normalized. Define

$$\mu(A) := \max_{i \neq j} \langle A_i, A_j \rangle$$

Good: μ is small.

- Predates compressed sensing.

Theory before compressed sensing

$$\min \left\| \boldsymbol{x}' \right\|_0 \quad \text{such that} \quad \boldsymbol{A}\boldsymbol{x}' = \boldsymbol{y} \tag{0.2}$$

Theorem (Donoho, Elad 2003)

Suppose that $\mu \cdot s < 1/2$. Then $\hat{\boldsymbol{x}} = \boldsymbol{x}$.

- Sparsity + incoherence leads to exact reconstruction of $\boldsymbol{x}$.

Challenge 1:
Challenge 2:

Theory before compressed sensing

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Challenge 1: Welch bound: Whenever $m < n/2$, $\mu > 1/\sqrt{2m}$.
$\Rightarrow$ the theorem requires $m \geq 2s^2$.
Challenge 2: Sparsity minimization takes exponential time. One needs to check each of the 2^m sparsity patterns.

ℓ_1 minimization (Chen, Donoho, Saunders 1999)

$$\min \left\|\boldsymbol{x}'\right\|_0 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x}' = \boldsymbol{y}. \tag{0.3}$$

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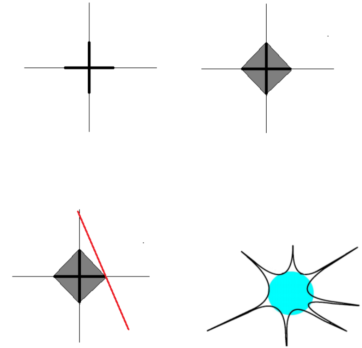

(a) Subspace hits corner of ℓ_1 ball

(b) ℓ_1 ball in high dimension

Theory before compressed sensing

$$\min \left\| \boldsymbol{x}' \right\|_1 \quad \text{such that} \quad \boldsymbol{A}\boldsymbol{x}' = \boldsymbol{y} \tag{0.4}$$

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Challenge 1: Welch bound: Whenever $m < n/2$, $\mu > 1/\sqrt{2m}$.
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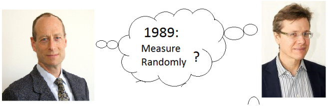
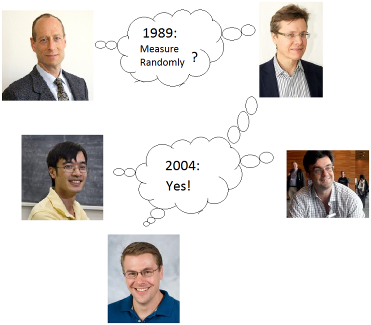
Compressed sensing ~2004



Compressed sensing $\sim$2004



Theorem (Candès, Romberg, Tao 2004, Donoho 2004)

$m = O(s\log(n))$ random Fourier coefficients are sufficient to reconstruct an s-sparse vector by ℓ_1 minimization.

- Ignoring logarithmic terms, $O(s)$ measurements are sufficient. $\log(n)$ is the 'price of not knowing the support'.
- Sets off an explosion of work still going today.

Back to MRI

Do the conditions of the theorem match MRI applications?

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Do the conditions of the theorem match MRI applications?
"If Bryce took a single breath, the image would be blurred. That meant deepening the anesthesia enough to stop respiration. It would take a full two minutes for a standard MRI to capture the image, but if the anesthesiologists shut down Bryce's breathing for that long, his glitchy liver would be the least of his problems."
[Wired Magazine 2010]
Compressed sensing + Vasanawala + Lustig = "Vasanawala and Lustig needed only 40 seconds to gather enough data to produce a crystal-clear image of Bryces liver [...] good enough for Vasanawala to see the blockages in both bile ducts. An interventional radiologist snaked a wire into each duct, gently clearing the blockages and installing tiny tubes that allowed the bile to drain properly. And with that a bit of math and a bit of medicine Bryces lab test results headed back to normal."

Theory of compressed sensing.

- Original theorem relied on constructing a *dual certificate* and analyzing the geometry of the ℓ_1 ball (challenging).
- Over time, simpler methods were developed.

Restricted isometry property (RIP) [Candès and Tao 2005]

Definition (Restricted isometry property)

We say that $\boldsymbol{A}$ satisfies the restricted isometry property of order s if there exists $\delta_s \in (0,1)$ such that

$$(1-\delta_s)\|\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{Ax}\|_2^2 \leq (1+\delta_s)\|\boldsymbol{x}\|_2^2$$

for all $\boldsymbol{x}$ satisfying $\|\boldsymbol{x}\|_0 \leq s$.

- Sparse signals are far from the null space of $\boldsymbol{A}$.
- Many kinds of *random matrices* satisfy the RIP with high probability as long as $m \geq O(s \text{ polylog}(n))$. No deterministic constructions are known!
- An appeal to *geometric functional analysis* or *probability in high dimensions*.

Implications of RIP

- $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$.
-
$$\hat{\boldsymbol{x}} := \arg\min \left\|\boldsymbol{x}'\right\|_1 \quad \text{such that} \quad \boldsymbol{A}\boldsymbol{x}' = \boldsymbol{y}$$

Theorem (Candès, Romberg, Tao 2005)

Suppose $\|\boldsymbol{x}\|_0 \leq s$ *and that* $\boldsymbol{A}$ *satisfies the restricted isometry property of order* $2s$ *with* $\delta_{2s} \leq \sqrt{2} - 1$*. Then*

$$\hat{\boldsymbol{x}} = \boldsymbol{x}.$$

- Robust to noise and approximate sparsity.
- Many other *greedy* reconstruction techniques are also exact under the RIP.

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Theorem (Candès, Romberg, Tao 2005)

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$$\hat{\boldsymbol{x}} = \boldsymbol{x}.$$

Proof sketch:

1. Note: $\boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \quad \Rightarrow \quad \boldsymbol{A}(\hat{\boldsymbol{x}} - \boldsymbol{x}) = 0.$
2. By the RIP, $\|\boldsymbol{A}(\hat{\boldsymbol{x}} - \boldsymbol{x})\|_2 \geq c\,\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|_2$. (Takes a bit of work.)

Thus $0 = \|\boldsymbol{A}(\hat{\boldsymbol{x}} - \boldsymbol{x})\|_2 \geq c\,\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|_2$
$\Rightarrow \hat{\boldsymbol{x}} - \boldsymbol{x} = 0 \quad \blacksquare$.

Restricted Isometry Property (RIP)

RIP theory (Candès, Tao 2005, Vershynin, Rudelson 2006):

- If $\boldsymbol{A}$ is a Gaussian matrix, then we need $m = O(s\log(n/s))$ measurements.
- If $\boldsymbol{A}$ is a subsampled discrete Fourier transform, we need $m = O(s\log^4(n))$ measurements.
- Open problem: Prove the RIP for the DFT with $m = O(s\log n)$ (solution would solve the Λ_1 problem).

Q: Is the RIP requirement too strong?

A: RIP gives a *uniform* result, i.e., based on $\boldsymbol{A}$ an adversary can pick $\boldsymbol{x}$, but whatever sparse $\boldsymbol{x}$ she picks, it will be reconstructed. Can we weaken our assumptions if the signal is not adversarial?

A general, RIPless theory (Candès, P. 2011)

We assume $\boldsymbol{a}_i \stackrel{\text{dist}}{=} \boldsymbol{a} \sim F$ and require:

- **Isotropy property:**
$$\mathbb{E}\, \boldsymbol{a}\boldsymbol{a}^* = \mathrm{Id}\,.$$
- **Incoherence property:**
$$\mu := \max_{1 \le i \le n} \langle \boldsymbol{a}, \boldsymbol{e}_i \rangle^2 < \infty \tag{0.5}$$

Note:

- The isotropy condition prevents A from being rank deficient when enough samples are taken. ($\lim_{m\to\infty} \frac{1}{m}\boldsymbol{A}^*\boldsymbol{A} = \mathrm{Id} \quad$ a.s.)
- Regardless of F, $\mu \ge 1$. Low μ implies strong results.

Bad example: Subsampled identity matrix

$\boldsymbol{a} \sim \sqrt{n} \cdot \mathrm{Uniform}(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \ldots, \boldsymbol{e}_n)$.

- **Isotropy property:**

$$\mathbb{E}\, \boldsymbol{a}\boldsymbol{a}^* = \frac{1}{n} \sum_{i=1}^{n} n e_i e_i^* = \mathrm{Id}\,.$$

- **Incoherence property:** $\mu := \sqrt{n}\, \langle e_1, \boldsymbol{e}_1 \rangle^2 = n \gg 1$

$$\boldsymbol{y} = \text{subset of the entries of } \boldsymbol{x}$$

$\Rightarrow$ Reconstruction of sparse $\boldsymbol{x}$ is impossible.

Measurement matrices that fit the model

- **Sensing vectors with independent components (with mean zero and variance 1).**
  - Gaussian ($\mu = 6 \log n$)
  - Binary ($\mu = 1$)
- **Subsampled orthogonal/unitary transforms.**
  - Fourier ($\mu = 1$)
- **Subsampled convolutions.** (The Fourier coefficients of the convolution vector must have magnitude (nearly) 1 to ensure (near) isotropy.)
- **Subsampled tight or continuous frames.**
  - Fourier sampled from continuous frequency spectrum.
- **Statistical linear model.**
  - Samples chosen independently from a population.

Theorem

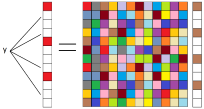
Theorem

Let x be an arbitrary fixed vector of length n with $\|x\|_0 = s$. Then with high probability $\hat{\boldsymbol{x}} = \boldsymbol{x}$ provided that $m \geq C \cdot \mu(F) \cdot s \cdot \log n$.

- Robust to noise and inexact sparsity.
- Prior stability results [Rudelson and Vershynin, 2008] for subsampled Fourier required $m \geq Cs \cdot \log^4(n)$ (using the RIP).

Near optimality

- **(Near) minimal number of measurements:** Take a_i to be a subsampled row from a complex DFT with $n = s \cdot t$. Take x to be the s-sparse dirac comb $\Rightarrow$ Fx is t-sparse.



It follows that $\langle a_i, x \rangle = 0$ for $i = 1, 2, \ldots, m$ with probability at least $1/n$ as long as $m \lesssim s \log n$, in which case no method can successfully recover x. (Note $\mu = 1$.)
This can be generalized to show that we need $m \gtrsim s\mu \log n$ when $\mu > 1$.

Statistical variable selection

Linear model in statistics

Example:

- $y_i :=$ lifespan of person i.
- $\boldsymbol{a}_i =$(exercise, weight, relationships, drugs, diet, ...)

Linear model:

$$y_i = \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + \boldsymbol{z}_i \quad i = 1, 2, \ldots, m.$$

Sparsity: Only a small number of *covariates* are significant.
Challenge: $\boldsymbol{a}_i$ usually does not satisfy the probabilistic assumptions needed for compressed sensing.

Sparsity with deterministic measurements

- What if $\boldsymbol{A}$ is non-random?
  No RIP. Adversarial $\boldsymbol{x}$ may be in the null space of $\boldsymbol{A}$.
- Can we say something about most signals $\boldsymbol{x}$?
- Solution: Put a prior distribution on $\boldsymbol{x}$. Prove that $\boldsymbol{x}$ is well reconstructed with high probability.

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- Solution: Put a prior distribution on $\boldsymbol{x}$. Prove that $\boldsymbol{x}$ is well reconstructed with high probability.
- **Prior:** The non-zero entries of $\boldsymbol{x}$ are chosen at random, and the signs of the non-zero entries of $\boldsymbol{x}$ are random.

Support recovery

- $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{z}$
- $\boldsymbol{z}_i \sim N(0, \sigma^2)$
- LASSO:

$$\hat{\boldsymbol{x}} := \arg\min \left\|\boldsymbol{A}\boldsymbol{x}' - \boldsymbol{y}\right\|_2^2 + 2\sigma\sqrt{\log n}\left\|\boldsymbol{x}'\right\|_1$$

Theorem

Let T be the support of $\boldsymbol{x}$ and suppose $m \geq |T| \log n$. Suppose that

$$\textbf{Assumption 1:}\ \min_{i \in T} |x_i| > 8\sigma\sqrt{2\log n}$$

$$\textbf{Assumption 2:}\ \mu(\boldsymbol{A}) \lesssim \frac{1}{\log n} \quad \textit{and} \quad ||A|| \lesssim \sqrt{\frac{n}{m}}.$$

Then with high probability $\hat{\boldsymbol{x}}$ has the same support as $\boldsymbol{x}$.

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Then with high probability $\hat{\boldsymbol{x}}$ has the same support as $\boldsymbol{x}$.

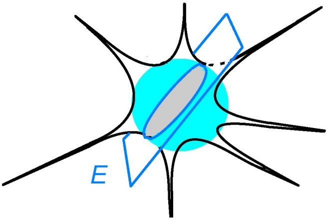
- There is also a near-optimal bound on $\|\boldsymbol{A}\hat{\boldsymbol{x}} - \boldsymbol{A}\boldsymbol{x}\|_2$.
- Problem: An optimal bound on $\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|_2$ without **Assumption 1**. (A sub-optimal bound is known (Dossal, Tesson 2012)).

Beyond compressed sensing

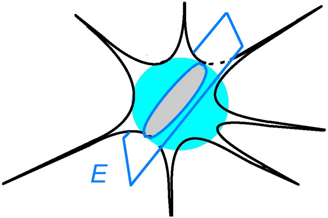
- **Theme 1: Subsampling.** The amount of information appears too small to reconstruct the signal.
- **Theme 2: Low dimensionality.** The signal resides in a set with small dimension in comparison to the ambient dimension.

General theory for Gaussian measurements

- Signal structure: $\boldsymbol{x} \in K \subset \mathbb{R}^n$.
- $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$.



General theory for Gaussian measurements



Theorem (M^* estimate: Milman 1981, Pajor, Tomczak 1985, Mendelson, Pajor, Tomczak 2007)

Consider a random subspace $\mathbb{E}$ in $\mathbb{R}^n$ of codimension m. Then with high probability

$$diam(K \cap E) \lesssim \frac{w(K)}{\sqrt{m}}.$$

- $w(K)$ is the *mean width* of K.

Matrix completion

$$\begin{pmatrix} M_{1,1} & M_{1,2} & M_{1,3} & M_{1,4} \\ M_{2,1} & M_{2,2} & M_{2,3} & M_{2,4} \\ M_{3,1} & M_{3,2} & M_{3,3} & M_{3,4} \\ M_{4,1} & M_{4,2} & M_{4,3} & M_{4,4} \end{pmatrix}, \quad M_{i,j} = \text{How much user } i \text{ likes movie } j$$

Matrix completion

$$\begin{pmatrix} ? & M_{1,2} & ? & M_{1,4} \\ M_{2,1} & ? & M_{2,3} & M_{2,4} \\ M_{3,1} & ? & M_{3,3} & M_{3,4} \\ ? & M_{4,2} & ? & M_{4,4} \end{pmatrix}, \quad M_{i,j} = \text{How much user } i \text{ likes movie } j$$

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Rank-1 model:

- $a_j =$ Amount of action in movie j
- $x_i =$ How much user i likes action

$$M_{i,j} = x_i \cdot a_j$$

Matrix completion

$$\begin{pmatrix} ? & M_{1,2} & ? & M_{1,4} \\ M_{2,1} & ? & M_{2,3} & M_{2,4} \\ M_{3,1} & ? & M_{3,3} & M_{3,4} \\ ? & M_{4,2} & ? & M_{4,4} \end{pmatrix}, \quad M_{i,j} = \text{How much user } i \text{ likes movie } j$$

Rank-1 model:

- a_j = Amount of action in movie j
- x_i = How much user i likes action

$$M_{i,j} = x_i \cdot a_j$$

Rank-2 model:

- b_j = Amount of comedy in movie j
- y_i = How much user i likes comedy

$$M_{i,j} = x_i \cdot a_j + y_i \cdot b_j$$

Example 2: Sparse binary data

Example: Predicting heart attacks (10 year study).

- $y_i = \begin{cases} 1 & \text{if patient } i \text{ has heart attack in 10 year window} \\ -1 & \text{otherwise} \end{cases}$
- $\boldsymbol{a}_i =$ (cholesterol, weight, BMI, blood-test results, exercise habits...).

Sparsity: Only a few factors are significant.
Which ones?

Riddle answer

Riddle: A sleeping dragon guards n locked treasure chests. s chests are filled with gold and the other $n-s$ are empty. You sneak into the dragon's lair carrying with you a large weighing machine. You wish to discover which chest holds the gold. How many times do you have to use the weighing machine?

Ans: Make $m = O(s\log(n/s))$ measurements of the weight of a random subset of chests. Reconstruct all weights using ℓ_1 minimization.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 100\text{ kg} \\ 0 \\ 100\text{ kg} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 200\text{ kg} \\ 100\text{ kg} \\ 0\text{ kg} \end{pmatrix}$$

Riddle answer

Ans: Make $m = O(s \log(n/s))$ measurements of the weight of a random subset of chests. Reconstruct all weights using ℓ_1 minimization.

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 100 \text{ kg} \\ 0 \\ 100 \text{ kg} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 200 \text{ kg} \\ 100 \text{ kg} \\ 0 \text{ kg} \end{pmatrix}$$

Q: Does the above matrix satisfy *any of* the assumptions of the theory we discussed?

A: No, but we can *precondition* the matrix by projecting onto the space orthogonal to the all ones vector.

Thank you!

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