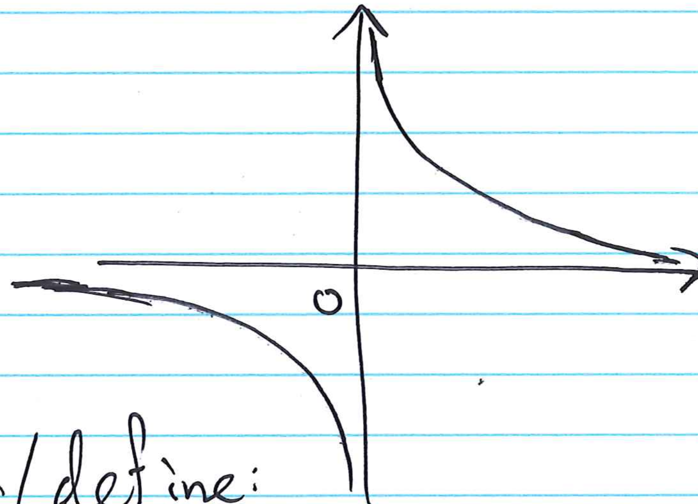
Limits involving Infinity.

Examples:

① $f(x) = \frac{1}{x}$

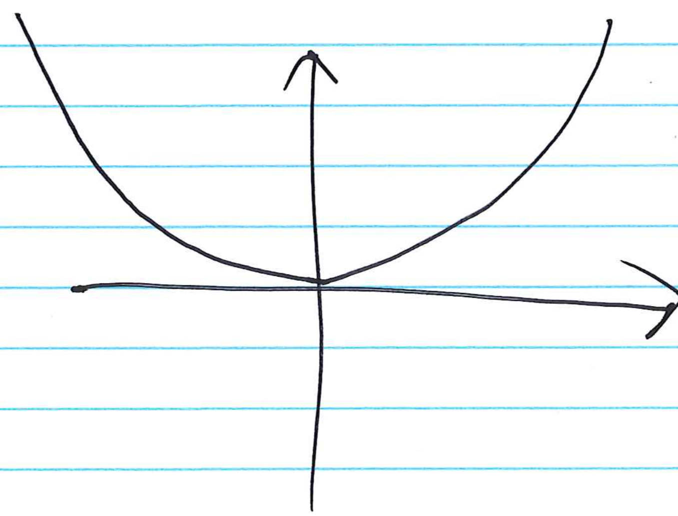


We want to discuss/explain/define:

$\lim_{x \to 0^+} f(x) = +\infty$, $\lim_{x \to 0^-} f(x) = -\infty$

$\lim_{x \to +\infty} f(x) = 0$, $\lim_{x \to -\infty} f(x) = 0$

② $f(x) = x^2$



$\lim_{x \to \infty} f(x) = +\infty$

$\lim_{x \to -\infty} f(x) = +\infty$

Definition:

* Suppose f is defined for all x near a with $x > a$. If $f(x)$ becomes arbitrarily large for x sufficiently close to a we write $\lim_{x \to a^+} f(x) = +\infty$.

$$\lim_{x \to a^-} f(x) = +\infty$$

$$\lim_{x \to a^+} f(x) = -\infty \iff \lim_{x \to a^+} (-f(x)) = +\infty.$$

$$\lim_{x \to a^-} f(x) = -\infty$$

* If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = +\infty$ we say that $\lim_{x \to a} f(x) = +\infty$, similarly for $-\infty$.

* If $\lim_{x \to a^\pm} f(x) = \pm\infty$, we say that $x = a$ is a vertical asymptote to $f(x)$.

__Examples:__ Find all vertical asymptotes of $f(x) = \dfrac{x-2}{(x-1)^2(x-3)}$

All vertical asymptotes should not be in the domain.

① Suspects: $x=1$, $x=3$.

②

		1		2		3	
$(x-2)$	$-$		$-$		$+$		$+$
$(x-1)^2$	$+$		$+$		$+$		$+$
$(x-3)$	$-$		$-$		$-$		$+$
$f(x)$	$+$		$+$		$-$		$+$

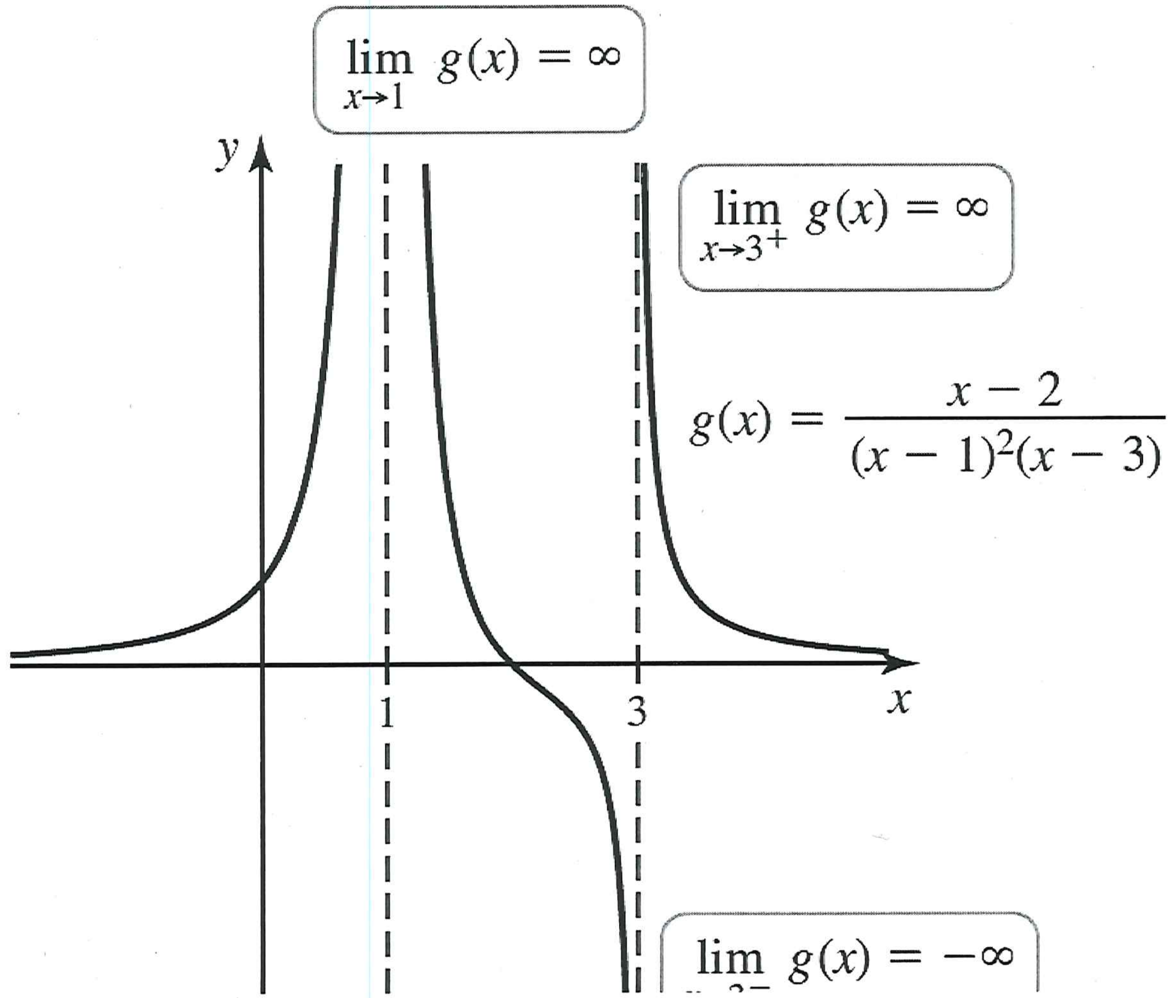
③ $x-2 \neq 0$ for $x=1,3$

$(x-1)^2(x-3)=0$ for $x=1,3$

$\lim\limits_{x\to 1^-} f(x) = +\infty$, $\lim\limits_{x\to 1^+} f(x) = +\infty$, $\lim\limits_{x\to 3^-} f(x) = -\infty$, $\lim\limits_{x\to 3^+} f(x) = +\infty$

④ Vertical asy. $x=1$ & $x=3$

Figure 2.27


$\lim_{x \to 1} g(x) = \infty$
$\lim_{x \to 3^+} g(x) = \infty$
$g(x) = \dfrac{x - 2}{(x - 1)^2(x - 3)}$
$\lim g(x) = -\infty$
y
x
1
3

(b) The same for $f(x) = \dfrac{x^2-4x+3}{x^2-1}$

$x^2-1=0 \rightsquigarrow x=\pm 1$

(1) Suspects $x=-1$, $x=+1$.

(2) $\left.\begin{aligned} x^2-4x+3 &\neq 0 \text{ at } x=-1 \\ x^2-1 &= 0 \text{ at } x=-1 \end{aligned}\right\} \Rightarrow x=-1$ ~~and the~~ is vertical asy.

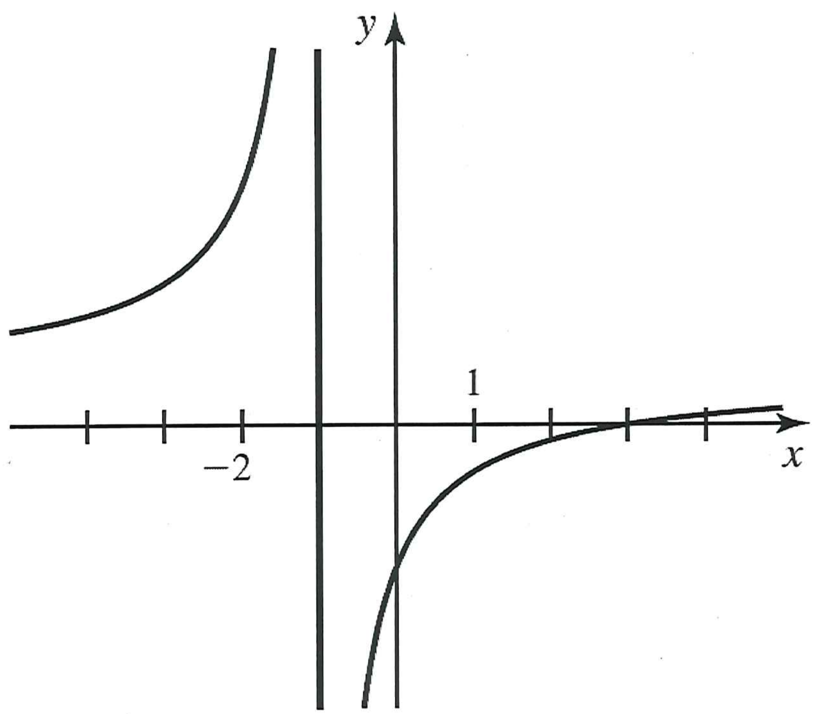
$\underline{x=+1:}$ $\dfrac{x^2-4x+3}{x^2-1} = \dfrac{(x-3)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = \dfrac{x-3}{x+1} \xrightarrow[x\to 1]{} \dfrac{-2}{2} = -1$

		-1		$+3$		~~$+1$~~	
$(x-3)$	$-$		$-$		$+$		$+$
$(x+1)$	$-$		$+$				
$f(x)$	$+$		$-$				

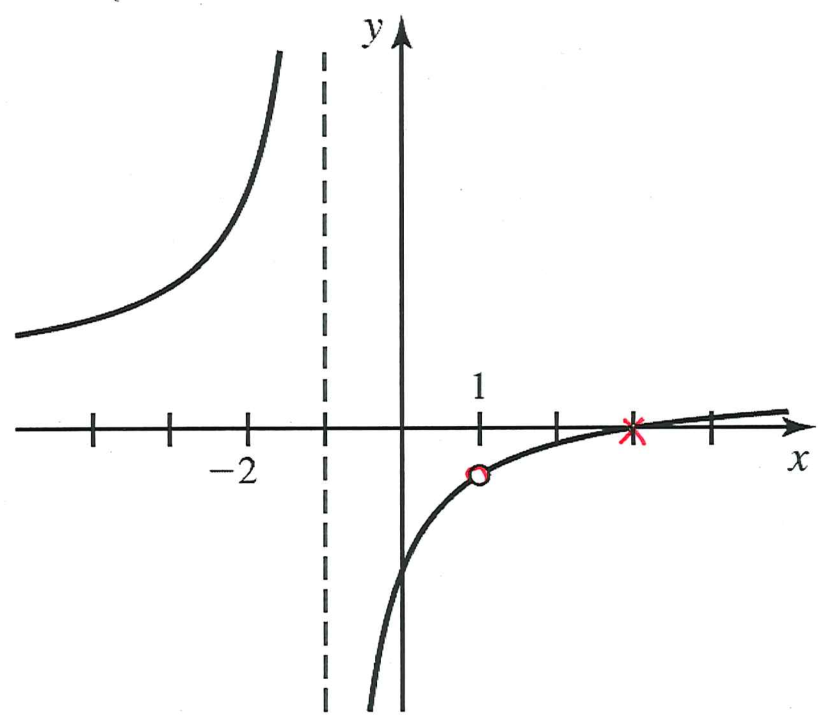
$\lim_{x\to -1^-} f(x) = +\infty$ $\qquad$ $\lim_{x\to -1^+} f(x) = -\infty.$

Figure 2.28

Two versions of the graph of $y = \dfrac{x^2 - 4x + 3}{x^2 - 1}$



Calculator graph



Correct graph

Definition:

If $f(x)$ becomes arbitrarily close to a number L for all sufficiently large and positive x, the we write $\lim_{x \to +\infty} f(x) = L$.

And we say that $y = L$ as a horizontal asymptote.

$$\lim_{x \to -\infty} f(x) = L \iff \lim_{x \to +\infty} f(-x) = L$$

Examples:

① $$\lim_{x \to +\infty} \frac{1}{x} = 0$$

$$\lim_{x \to +\infty} \frac{1}{x^2} = 0$$

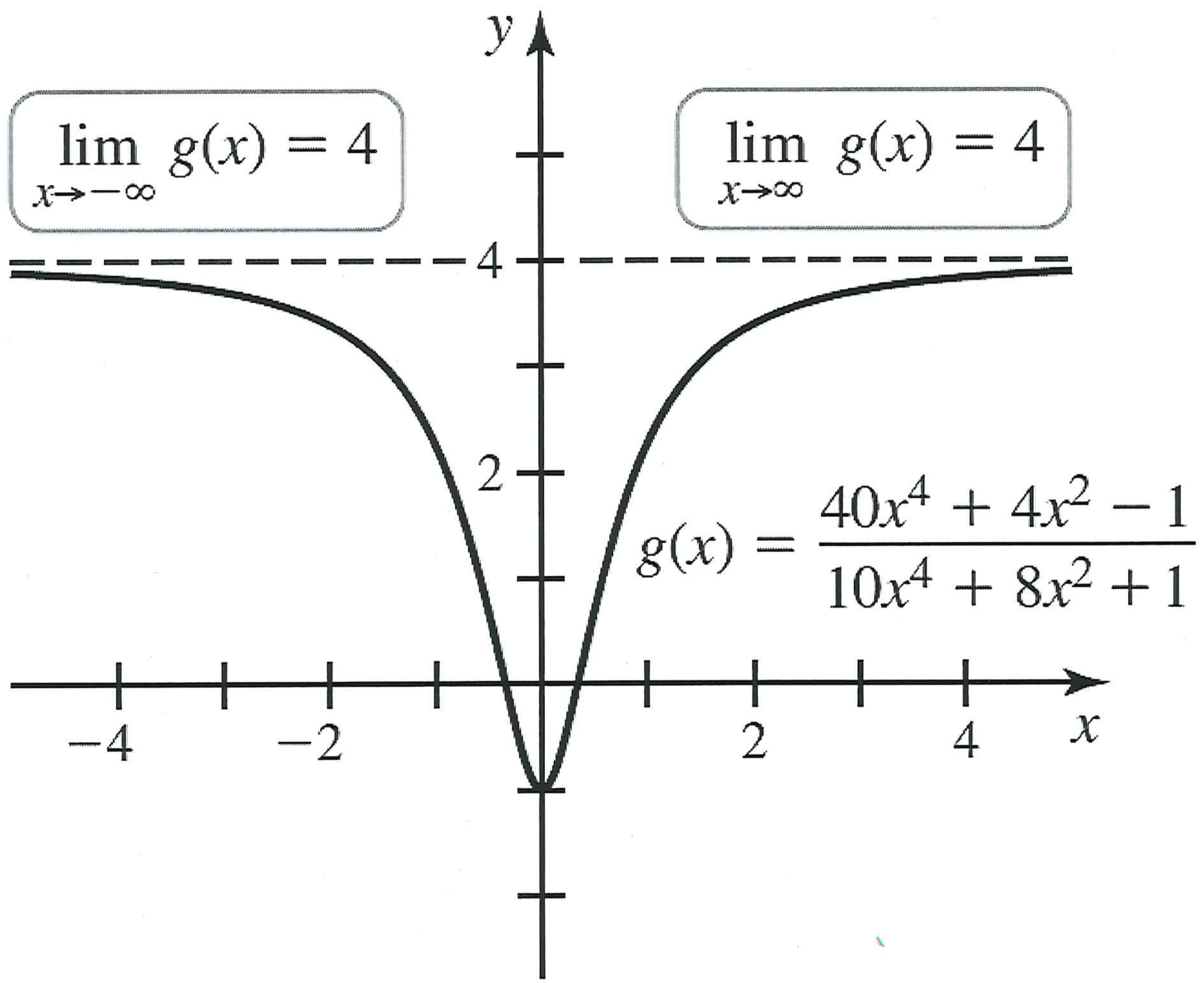
$$\vdots$$

$$\lim_{x \to +\infty} \frac{1}{x^n} = 0 \qquad n > 0.$$

② $$\lim_{x \to \infty} \frac{40x^4 + 4x^2 - 1}{10x^4 + 8x^2 + 1} = \lim_{x \to +\infty} \frac{\cancel{x^4}\left(40 + \frac{4}{x^2} - \frac{1}{x^4}\right)}{\cancel{x^4}\left(10 + \frac{8}{x^2} + \frac{1}{x^4}\right)}$$

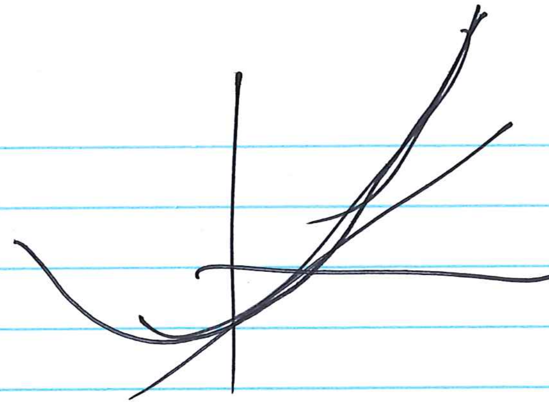
$$= \lim_{x \to +\infty} \frac{40 + \overbrace{\frac{4}{x^2}}^{\to 0} - \overbrace{\frac{1}{x^4}}^{\to 0}}{10 + \underbrace{\frac{8}{x^2}}_{\to 0} + \underbrace{\frac{1}{x^4}}_{\to 0}} = \frac{40}{10} = 4$$

Figure 2.39



y
$\lim_{x \to -\infty} g(x) = 4$
$\lim_{x \to \infty} g(x) = 4$
4
2
$g(x) = \dfrac{40x^4 + 4x^2 - 1}{10x^4 + 8x^2 + 1}$
−4
−2
2
4
x

③ $\lim\limits_{x\to+\infty} \dfrac{3x+2}{x^2-1}$



$= \lim\limits_{x\to+\infty} \dfrac{\cancel{x}(3+\frac{2}{x})}{x^{\cancel{2}}(1-\frac{1}{x^2})} = \lim\limits_{x\to+\infty} \dfrac{3+\frac{2}{x}}{x(1-\frac{1}{x^2})} = 0$

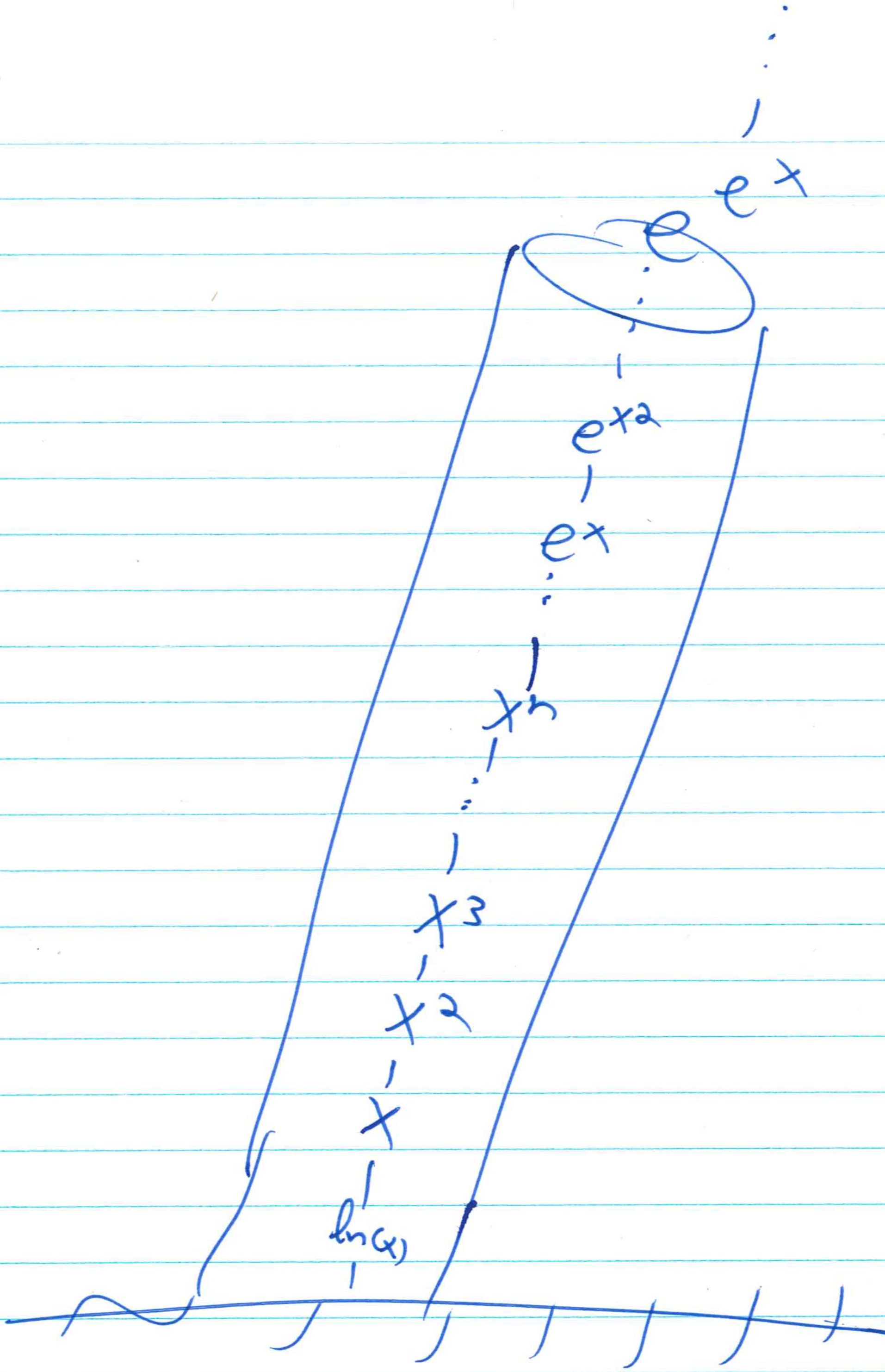
$\sim \dfrac{3}{x}$

④ $\lim\limits_{x\to+\infty} \dfrac{x^n}{e^x} = 0$

⑤ $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x - 1{,}000{,}000{,}000{,}000} = 0$

⑥ $\lim\limits_{x\to+\infty} \dfrac{x^3-2x+1}{2x+4} \sim \lim\limits_{x\to+\infty} \dfrac{x^3}{2x} = \lim\limits_{x\to+\infty} \dfrac{x^2}{2} = +\infty$

$\lim\limits_{x\to-\infty} \dfrac{x^3-2x+1}{2x^2+4} \sim \lim\limits_{x\to-\infty} \dfrac{x^3}{2x^2} = \lim\limits_{x\to-\infty} \dfrac{x}{2} = -\infty$



$$\lim_{x \to \infty} \frac{e^x}{e^{x^2}} = 0$$

$$\lim_{x \to \infty} \frac{\ln(x)}{x^\alpha} = 0$$

$\alpha > 0$