We play with the singular value decomposition (svd). You may find the svd in fourth or fifth editions of your textbook (but unfortunately, not third), which is available at UBC library.

1. Let $Q, W \in \mathbb{R}^{5 \times 5}$ be orthogonal matrices. Let $q_1, q_2, q_3, q_4, q_5$ be the columns of $Q$ and $w_1, w_2, w_3, w_4, w_5$ be the columns of $W$. To be precise $q_i$ is the $i$th column of $Q$ and $w_i$ is the $i$th column of $W$. Give a basis for the null space of the following matrices, which are almost in SVD form:

(a) (5 pts)

$$A = Q\Sigma W^T,$$

$$\Sigma = \begin{bmatrix}
10 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

**Solution:** We put this into the form of the standard SVD. First, we express $Q, \Sigma, W$ using the canonical basis vectors. We have

$$Q = \sum q_i e_i^T, \quad W = \sum w_i e_i^T, \quad \Sigma = 10 e_1^T + 2 e_3^T.$$ 

We then take the product of the three (with many cancelations since canonical basis vectors are orthonormal):

$$A = Q\Sigma W^T = 10 q_1 w_1^T + 2 q_3 w_3^T.$$ 

This is a form of the SVD (rank-1 decomposition). As seen in class, the columns of $W$ that are left out give an orthonormal basis for the null space. Thus, the solution is $w_2, w_4, w_5$.

(b) (5 pts)

$$B = QSW^T,$$

$$S = \begin{bmatrix}
0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & -5 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

**Solution:** Use the same steps as above, but with

$$S = 3 e_1^T e_4^T - 5 e_2^T e_5^T.$$ 

Thus,

$$A = 3 q_1 w_4^T + 5(-q_1) w_5^T.$$ 

Once again, this is SVD form. The solution is $w_1, w_2, w_3$.

2. Let $A$ be a matrix with SVD $A = U\Sigma V^T$ (this is the standard SVD, not the reduced SVD). Suppose that

$$\Sigma = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
\end{bmatrix}.$$
A right-inverse of $A$ is a matrix $B$ which satisfies $AB = I$. Does $A$ have two or more right inverses? If so, find two right inverses. If not, why not?

Hint: You may write your answer to the above question in terms of $U$ and $V$. For example, if the question had been, what is $A^T A$? Then the answer would have been

$$A^T A = V \Sigma^T \Sigma V^T = V \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} V^T.$$

**Solution:** Let $B$ be a right inverse. Since $U$ and $V$ are invertible, we may write $B = V S U^T$ for some matrix $S \in \mathbb{R}^{4 \times 3}$. Then we have

$$I = AB = U \Sigma V^T V S U^T.$$

We can massage the above equation to give

$$I = \Sigma S.$$

Any solution to the above equation gives a right inverse, i.e., any matrix of the form $V S U^T$, where

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \\ a & b & c \end{bmatrix}, \quad a, b, c \in \mathbb{R}$$

3. (10 pts) (This question is inspired by the idea of noise shaping – the fact that in some applications one can force the noise to take a certain pattern.) Suppose that you have data from the (noiseless) linear model, but it becomes corrupted in the following way: One fixed, but unknown, constant is added to each entry of the data. To be precise, assume the following model:

$$y = Ax + z, \quad z = \begin{bmatrix} c \\ c \\ c \end{bmatrix}.$$

The matrix $A$ has the singular value decomposition

$$A = U \Sigma V^T, \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad V \in \mathbb{R}^{3 \times 3} \text{ is orthogonal.}$$

You observe $y$ and $A$, but not $c$ or $x$.

Is there a linear way to determine $x$? To be precise, is there a matrix $W$ satisfying $W y = x$ no matter what $c$ is? If so, determine the matrix $W$ (the answer could be written in terms of $V$).
**Solution:** Yes! We need to find a left inverse that has $z$ in the null space so $Wy = W(Ax+z) = WAx = x$.

Following the ideas of the last question, any matrix $W$ of the form

$$W = VSU^T, \quad S = \begin{bmatrix} 1/3 & 0 & 0 & a \\ 0 & 1/2 & 0 & b \\ 0 & 0 & 1 & d \end{bmatrix}$$

is a left inverse. We will choose $a, b, d$ so that $Wz = 0$ (regardless of $c$), i.e., we need $[1, 1, 1, 1]^T$ to be in the null space of $W$. We have

$$Wz = cV \begin{bmatrix} 1/3 + a \\ 1/2 + b \\ 1 + d \end{bmatrix}$$

Thus, we pick $a = -1/3, b = -1/2, d = -1$. The solution is

$$W = VSU^T, \quad S = \begin{bmatrix} 1/3 & 0 & 0 & -1/3 \\ 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$