1. Solve the differential equation given the initial conditions $x_1(0) = -3$, $x_2(0) = 4$.

$$\frac{d}{dt}x_1(t) = c_2 \sin(t) + x_2(t)$$

$$\frac{d}{dt}x_2(t) = -2x_1(t) - 2x_2(t)$$

**Solution:**

You compute

$$\det(\begin{bmatrix} 0 - \lambda & 1 \\ -2 & -2 - \lambda \end{bmatrix}) = \lambda^2 + 2\lambda + 2$$

and find the roots as $-1+i$ and $-1-i$ (complex eigenvalues come in conjugate pairs because the matrix has real entries)

$$\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ 1 \end{bmatrix}$$

So we have

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{(-1+i)t} \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ 1 \end{bmatrix} + e^{(-1-i)t} \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}i \\ 1 \end{bmatrix}$$

We use the initial conditions to solve for $c_1, c_2$ in

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{(-1+i)t} \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ 1 \end{bmatrix} + c_2 e^{(-1-i)t} \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}i \\ 1 \end{bmatrix}$$

So we have

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}i \\ 1 \end{bmatrix}$$

from which we obtain $c_1$, $c_2$ using

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}i \\ 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} i \frac{1}{2} + \frac{1}{2}i \\ -i \frac{1}{2} + \frac{1}{2}i \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Thus $c_1(t) = 2 - i$ and $c_2 = 2 + i$. We then substitute to obtain

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = (2 - i)e^{(-1+i)t} \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}i \\ 1 \end{bmatrix} + (2 + i)e^{(-1-i)t} \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}i \\ 1 \end{bmatrix}$$

Thus for example the first term, top entry is

$$(2 - i)e^{-t}(\cos(t) + i\sin(t))(-\frac{1}{2} - \frac{1}{2}i)$$

$$= e^{-t} \left( (-\frac{3}{2} \cos(t) + \frac{1}{2} \sin(t)) + i(-\frac{1}{2} \cos(t) - \frac{3}{2} \sin(t)) \right)$$

The other term is the complex conjugate and so we obtain the solution $x_1(t) = e^{-t}(-3 \cos(t) + \sin(t))$. Similarly the first term, bottom entry is

$$(2 - i)e^{-t}(\cos(t) + i\sin(t)) = e^{-t}((2 \cos(t) + \sin(t)) + i(2 \sin(t) - \cos(t)))$$

The other term is the complex conjugate and so we obtain the solution $x_2(t) = e^{-t}(4 \cos(t) + 2 \sin(t))$. You can verify that these satisfy the differential equation (no complex numbers required!) $\frac{d}{dt}e^{-t}(-3 \cos(t) + \sin(t)) = e^{-t}(3 \sin(t) + \cos(t) + 3 \cos(t) - \sin(t)) = e^{-t}(4 \cos(t) + 2 \sin(t)) = x_2(t)$ and $\frac{d}{dt}e^{-t}(4 \cos(t) + 2 \sin(t)) = e^{-t}(-4 \sin(t) + 2 \cos(t) - 4 \cos(t) - 2 \sin(t)) = e^{-t}(-2 \cos(t) - 6 \sin(t)) = -2(-3 \cos(t) + \sin(t)) - 2(4 \cos(t) + 2 \sin(t)) = -2x_1(t) - 2x_2(t)$. It would be easier to do this by computing the Real and Imaginary part of our solution (what $c_1$ multiplies) and then proceeding to solve for $c_1, c_2$. 
2. You are given a 3 dimensional vector space \( V \subseteq \mathbb{R}^5 \). Could there be a \( 3 \times 6 \) matrix \( A \) with nullspace of \( A \) being \( V \)? Explain. Could there be \( 6 \times 5 \) matrix \( B \) with nullspace of \( B \) being \( V \)? Explain. In either case, if you were given a basis for the three dimensional space \( V \), how would you find the desired matrix assuming it exists.

Solution:
In the first case nullspace(\( A \)) \( \subseteq \mathbb{R}^6 \) while \( V \subseteq \mathbb{R}^5 \) so this can never work. In the second case we want \( \dim(\text{nullspace}(B)) = 3 = 5 - \text{rank}(B) \) and so \( \text{rank}(B) = 2 \). We have that \( \dim(V^\perp) = 5 - 3 = 2 \) and so choose a basis for \( V^\perp \), say \( x, y \). Then you could form \( B \) with 4 rows equal to \( x^T \) and one row equal to \( y^T \).

3. Consider the two planes \( \pi_1: x - y + 2z = 3 \) and \( \pi_2: x + 2y + 3z = 6 \).

a) Find the intersection of \( \pi_1 \) and \( \pi_2 \) in vector parametric form.

b) What is the angle (or just the cosine of the angle) formed by the two planes? (This is defined as the angle between their normal vectors. A normal vector is a vector orthogonal \( u - v \) for every \( u, v \) in the plane.)

c) Find the distance of the point \((-1, 2, 2)\) to the plane \( \pi_1 \). (That is, find the distance between \((-1, 2, 2)\) and the closest point in \( \pi_1 \).)

d) Find the equation of the plane parallel to \( \pi_1 \) through the point \((3, 2, 0)\).

e) Imagine the direction \((0, 0, 1)^T\) as pointing straight up from your current position \((0, 0, 0)^T\) in 3-space and the plane \( \pi_2 \) as a physical plane. If a marble is placed on \( \pi_2 \) at the point \((6, 0, 0)^T\), what direction will the marble roll under the influence of gravity?

Solution:

b) What is the angle (and cosine of the angle) formed by the two planes? This is most readily computed at the angle between the normals and hence \( \cos \theta = (1, -2, 2)^T \cdot (1, 2, 3) / \| (1, -2, 2)^T \| \| (1, 2, 3)^T \| = 5/(2\sqrt{21}) \). We compute \( \theta = \cos^{-1}(5/(2\sqrt{21})) \approx 57^\circ \).

c) Find the distance of the point \((-1, 2, 2)\) to the plane \( \pi_1 \). We first note that \((3, 0, 0)^T \in \pi_1 \) and then the length of the projection of \((-4, 2, 2)^T = (-1, 2, 2)^T - (3, 0, 0)^T \) onto the normal \((1, -1, 2)^T \), which is \( \frac{2^2}{6} (1, -1, 2)^T \) and so the distance is \( \frac{2}{3} \sqrt{6} \).

d) Find the equation of the plane parallel to \( \pi_1 \) through the point \((3, 2, 0)\). We simply take the plane to be \( x - y + 2z = c \) for some constant \( c \) and since \((3, 2, 0)\) is on the plane we take \( c = 1 \) so that the equation of the plane is \( x - y + 2z = 1 \).

e) Imagine the direction \((0, 0, 1)^T\) as pointing straight up from your current position \((0, 0, 0)^T\) in 3-space and the plane \( \pi_2 \) as a physical plane. If a marble is placed on \( \pi_2 \) at the point \((6, 0, 0)^T\), what direction will the marble roll under the influence of gravity? There are a variety of ways to approach this problem. Using forces, we know that the marble is acted upon by two forces: gravity in the direction \((0, 0, -1)^T \) and force upward by the plane in the direction of the normal \((1, 2, 3)^T \). The resultant force is in the intersection of the plane formed by these two vectors and \( \pi_1 \) where we take the \( z \) coordinate negative since we will be moving downhill. Alternatively, we can project the gravity vector \((0, 0, -1)\) into the plane by
subtracting the projection of the vector $(0, 0, -1)^T$ onto the normal:

$$
\begin{bmatrix}
0 \\
0 \\
-1
\end{bmatrix} - \frac{3}{14}
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} = 
\begin{bmatrix}
3/14 \\
6/14 \\
-5/14
\end{bmatrix}
$$

and so the marble rolls in the direction $(3, 6, -5)^T$.

4. Given a matrix $A \in \mathbb{R}^{n \times n}$, we define the trace

$$
\text{tr}(A) = \sum_{i=1}^{n} A_{i,i},
$$

i.e., the sum of the diagonal. This is an important quantity.

a) Let $A, B \in \mathbb{R}^{n \times n}$. Show that $\text{tr}(AB) = \text{tr}(BA)$. Hint: You may wish to express $AB$ using the dot products between rows of $A$ and columns of $B$. To be precise, let $u_1, u_2, ..., u_n$ be the columns of $A^T$ (rows of $A$) and $v_1, v_2, ..., v_n$ be the columns of $B$. Then $(AB)_{i,j} = u_i \cdot v_j$. You can then show that $\text{tr}(AB)$ is the dot product between $A^T$ and $B$ (it’s up to you to define this dot product between matrices).

Solution:

We have

$$
\text{tr}(AB) = \sum_{i} u_i \cdot v_i = \sum_{i,j} (A^T)_{i,j} B_{i,j} = \sum_{i,j} (B^T)_{i,j} A_{i,j} = \text{tr}(BA).
$$

In words, we see that the trace of $AB$ is the ‘dot product’ between the matrices $A^T$ and $B$ (i.e., expand $A^T$ and $B$ into vectors and take the dot products). In the third equality above, we used the fact that the ‘dot product’ between $A^T$ and $B$ is the same as the ‘dot product’ between $B^T$ and $A$.

b) Suppose that $A$ can be diagonalized as $A = MDM^{-1}$ where $D$ is a diagonal matrix of eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Show that

$$
\text{tr}(A) = \sum_{i=1}^{n} \lambda_i.
$$

Solution:

By part $A$, we may move $M^{-1}$ next to $M$ and cancel:

$$
\text{tr}(A) = \text{tr}(MDM^{-1}) = \text{tr}(M^{-1}MD) = \text{tr}(D) = \sum_{i=1}^{n} \lambda_i.
$$

Important note: The above equality is true even if $A$ cannot be diagonalized. In other words, let $\lambda_1, \lambda_2, ..., \lambda_n$ be the $n$ solutions to the characteristic equation $\det(A - \lambda I) = 0$. By the Fundamental Theorem of Algebra, there are always $n$ solutions when counted with multiplicity. These are the eigenvalues of $A$. Then

$$
\text{tr}(A) = \sum_{i=1}^{n} \lambda_i.
$$

You may use this fact without proof.
5. Let $A$ be a $n \times n$ matrix of real entries satisfying $A^2 = -I$. Show that
   a) $A$ is invertible (or nonsingular)
   b) $A$ has no real eigenvalues
   c) $n$ is even
   d) (harder question) $\det(A) = 1$. (Hint: Try using the previous question.)

Solution:

a) $A$ is nonsingular because $AA = A^2 = -I$ and so $(-A)(A) = -A^2 = -(-I) = I$ and so $A^{-1} = -A$.

b) $A$ has no real eigenvalues. If $Ax = \lambda x$ for some $x \neq 0$, then $A^2x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda^2x$. At the same time $A^2x = (-I)x = -x$. Using $x \neq 0$. We deduce $\lambda^2 = -1$ and so there are no real values for $\lambda$. In fact if $\lambda$ is an eigenvalue then $\lambda^2 = 1$ and so $\lambda = i$ or $-i$.

c) $n$ is even. We compute $\det(A^2) = (\det(A))^2 = \det(-I) = (-1)^n$. We deduce that $n$ is even. Otherwise if $n$ is odd we have $(\det(A))^2 = -1$ at the same time $\det(A)$ is real, which is a contradiction.

d) We immediately have $\det(A) = \pm 1$, from $(\det(A))^2 = 1$. But we need to show that $\det(A) = 1$.

There are two extra facts we can use. The product of the eigenvalues, using multiplicities of the roots, is $\det(A)$, and the sum of the eigenvalues is $\text{tr}(A)$. The eigenvalues are $i$ or $-i$, by our argument in (b). Let the multiplicity of $i$ as a root of $\det(A - \lambda I)$ be $s$ and let the multiplicity of $-i$ be $t$, so that $s + t = n$ (the polynomial $\det(A - \lambda I)$ will factor into linear factors over $\mathbb{C}$). The sum of the eigenvalues is $(s - t)i$. Thus $\text{tr}(A) = (s - t)i$. But the trace of $A$ is real, since the entries of $A$ are real, so $s = t$. But then $\det(A) = (i)^s(-i)^t = (1)^s = 1$ using the determinant as the product of the eigenvalues counted with multiplicity.

6. Consider two vectors spaces $U, V$, subspaces of $\mathbb{R}^m$. Define $U + V = \{ u + v : u \in U, v \in V \}$. (This is called the Minkowski sum.) Show that $U + V$ is a vector space. Now show that $\dim(U) + \dim(V) = \dim(U \cap V) + \dim(U + V)$.

(Hint: if we have an $m \times n$ matrix $A$ then $n = \dim(\text{nullspace}(A)) + \text{rank}(A)$. How should we form $A$? )

Solution:

We verify that $U + V$ is a vector space by verifying closure. Let $w_1, w_2 \in U + V$ with $w_1 = u_1 + v_1$ and $w_2 = u_2 + v_2$. Then $w_1 + w_2 = u_1 + v_1 + u_2 + v_2 = (u_1 + u_2) + (v_1 + v_2)$. Since $U, V$ are vector spaces, then $u_1 + u_2 \in U$ and $v_1 + v_2 \in V$. Then $w \in U + V$. Similarly $kw = k(u_1 + v_1) = ku_1 + kv_1$.

Consider $U \cap V$. If $U \cap V = \{0\}$ then Let $W = \{ w_1, w_2, \ldots, w_s \}$ be a basis for $U \cap V$. We can extend $\{w_1, w_2, \ldots, w_s\}$ to a basis $U = \{u_1, u_2, \ldots, u_k\}$ for $U$ and to a basis $V = \{v_1, v_2, \ldots, v_t\}$ for $V$ where $W \subseteq U$ and $W \subseteq V$. Now we claim $U \cup V = (U \backslash W) \cup W \cup (V \backslash W)$ is a basis for $U + V$. Given that the new set contains a basis for $U$ and a basis for $V$ we deduce that $U + V \subseteq \text{span}(U \cup V)$. Now is $U \cup V$ an independent set? For convenience assume the first $s$ vectors of $U$ and $V$ are $W$ so that $U = \{w_1, \ldots, w_s, u_{s+1}, u_{s+2}, \ldots, u_k\}$ and
\[ \mathcal{V} = \{ \mathbf{w}_1, \ldots, \mathbf{w}_s, \mathbf{v}_{s+1}, \mathbf{v}_{s+2}, \ldots, \mathbf{v}_\ell \}. \] Assume

\[ \sum_{i=1}^{s} c_i \mathbf{w}_i + \sum_{i=s+1}^{k} a_i \mathbf{u}_i + \sum_{i=s+1}^{\ell} b_i \mathbf{v}_i = 0. \]

Following from the linear independence of \( \mathcal{U} \) we deduce that if \( b_{s+1} = b_{s+2} = \cdots = b_\ell = 0 \), then \( c_1 = c_2 = \cdots = c_s = a_{s+1} = a_{s+2} = \cdots = a_k = 0 \). So we may assume \( b_j \neq 0 \) for some \( j \in \{ s+1, s+2, \ldots, \ell \} \). Similarly we may deduce that \( a_p \neq 0 \) for some \( p \in \{ s+1, s+2, \ldots, k \} \). But now \( \sum_{i=s+1}^{\ell} b_i \mathbf{v}_i \neq 0 \) (else we violate the linear independence of \( \mathcal{V} \)) and if we set

\[ \mathbf{u} = \sum_{i=1}^{s} c_i \mathbf{w}_i + \sum_{i=s+1}^{k} a_i \mathbf{u}_i + \sum_{i=s+1}^{\ell} b_i \mathbf{v}_i \quad \text{and} \quad \mathbf{v} = \sum_{i=s+1}^{\ell} b_i \mathbf{v}_i \neq 0 \]

we have \( \mathbf{u} = -\mathbf{v} \neq 0 \). But \( -\mathbf{u} \in \mathcal{U} \) and \( \mathbf{v} \in \mathcal{V} \) and so \( \mathbf{v} \notin \mathcal{U} \cap \mathcal{V} \). But \( \mathbf{v} \in \text{span} \mathcal{V} \setminus \mathcal{W} \) and so \( \mathbf{v} \notin \mathcal{U} \cap \mathcal{V} \), a contradiction. Thus we deduce that \( (\mathcal{U} \setminus \mathcal{W}) \cup \mathcal{W} \cup (\mathcal{V} \setminus \mathcal{W}) \) are linearly independent. This yields \( \dim(\mathcal{U} + \mathcal{V}) = k + \ell - s \) where \( \dim \mathcal{U} \cap \mathcal{V} = s \). Hence

\[ \dim(\mathcal{U}) + \dim(\mathcal{V}) = \dim(\mathcal{U} \cap \mathcal{V}) + \dim(\mathcal{U} + \mathcal{V}). \]