1. Let $A \in \mathbb{R}^{m \times n}$ have rank 1. Show that there exist non-zero vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ so that $A = xy^T$. (Hint: Try a simple case and also compute $xy^T$ for some simple choices for $x$ and $y$.) (Comment: You could explore how to generalize such a result to higher rank.)

**Solution:**

If $A$ has rank 1 then $\dim(\text{colsp}(A)) = 1$ and so there is a basis for the column space consisting of a single vector $x \in \mathbb{R}^m$. Now each column of $A$ is a multiple of $x$ and let the $j$th column be $y_j x$. Now letting $y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$, we obtain $A = xy^T$. The condition is an ‘if and only if’. If $A = xy^T$, where $x, y$ are nonzero vectors, then $\text{colsp}(A) = \text{span}(x)$ and so $\text{rank}(A) = 1$.

2. Determine bases for the following subspaces of $\mathbb{R}^3$.

a) the line $x = 5t, y = -2t, z = t$.

b) all vectors of the form $(a, b, c)^T$ such that $a - 3b = 2c$.

**Solution:**

a) the line $x = 5t, y = -2t, z = t$ has basis

$$\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}.$$  

b) all vectors of the form $(a, b, c)^T$ such that $a - 3b = 2c$ has the basis

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$ or other choices.

3. Let $A = \begin{bmatrix} 0 & 1 & 1 & 2 & -3 & 1 \\ 0 & 2 & 0 & 6 & -6 & 0 \\ 0 & 3 & 7 & 2 & -9 & 7 \\ 0 & 2 & 2 & 4 & -4 & 3 \end{bmatrix}$.

Determine a basis for the column space of $A$ (chosen from columns of $A$) and determine a basis for the row space of $A$. Also give a basis for the nullspace of $A$, namely $\{x \in \mathbb{R}^6 : Ax = 0\}$.

**Solution:**

We have

$$\begin{bmatrix} 0 & 1 & 1 & 2 & -3 & 1 \\ 0 & 2 & 0 & 6 & -6 & 0 \\ 0 & 3 & 7 & 2 & -9 & 7 \\ 0 & 2 & 2 & 4 & -4 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 & 2 & -3 & 1 \\ 0 & 0 & -2 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for the column space of $A$ is

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 7 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ -9 \\ -4 \end{bmatrix}.$$
and a basis for the row space of $A$ is

$$(0, 1, 1, 2, -3, 1)^T, \quad (0, 0, -2, 2, 0, -2)^T, \quad (0, 0, 0, 2, 1)^T$$

A basis for the nullspace of $A$ is

$$\begin{pmatrix} 0 \\ -3/2 \\ -1 \\ 0 \\ -1/2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In all cases, other bases are possible.

4. Show that the set of all vectors $(b_1, b_2, b_3, b_4)^T$ such that the system below is consistent (i.e. can be solved)

$$\begin{pmatrix} 2 & 3 & 1 \\ 4 & 3 & 3 \\ 1 & 3 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

is a subspace of $\mathbb{R}^4$. Then find a basis of the subspace.

Solution:

Let $V = \{ \mathbf{b} : A\mathbf{x} = \mathbf{b} \text{ is consistent } \}$. Then if $\mathbf{u}, \mathbf{v} \in V$, we have that there exists a $\mathbf{y}$ with $A\mathbf{y} = \mathbf{u}$ and there exists a $\mathbf{z}$ with $A\mathbf{z} = \mathbf{v}$. But then $A(\mathbf{y} + \mathbf{z}) = A\mathbf{y} + A\mathbf{z} = \mathbf{u} + \mathbf{v}$ and hence $\mathbf{u} + \mathbf{v} \in V$. Similarly $A(k\mathbf{y}) = kA\mathbf{y} = k\mathbf{u}$. Hence $k\mathbf{u} \in V$ as well and so $V$ is a vector space. An easier proof is that $\mathbf{b} \in V$ if and only if $\mathbf{b} \in \text{colsp}(A)$ and we already know that the $\text{colsp}(A)$ is a vector space (it is the span of the columns of $A$).

$$\begin{pmatrix} 2 & 3 & 1 & b_1 \\ 4 & 3 & 3 & b_2 \\ 1 & 3 & 0 & b_3 \\ 2 & 0 & 2 & b_4 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 3 & 1 & b_1 \\ 0 & -3 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & -3/2b_1 + 1/2b_2 + b_3 \\ 0 & 0 & 0 & b_1 - b_2 + b_4 \end{pmatrix}$$

The system is consistent if $-b_1 + 1/2b_2 + b_3 = 0$ and $b_1 - b_2 + b_4 = 0$. We now solve this system in $b_1, b_2, b_3, b_4$ to obtain

$$\begin{pmatrix} -3/2 & 1/2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -3/2 & 1/2 & 1 & 0 \\ 0 & -2/3 & 2/3 & 1 \end{pmatrix}$$

and so the basis is

$$\begin{pmatrix} 1/2 \\ 3/2 \\ 1/0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ or other choices}$$

I mention other choices and in this case it would be slightly easier to identify $b_3, b_4$ as the pivot variables and $b_1, b_2$ as the free variables.

Again, a different and easier solution is to identify the set of possible $\mathbf{b}$ as $\text{colsp}(A)$ and then, by Gaussian Elimination, a basis is

$$\begin{pmatrix} 2 \\ 4 \\ 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 3 \\ 3 \\ 0 \end{pmatrix}$$
5. Let $A$ be an $n \times n$ matrix with various eigenvalues including $\lambda$ and $\mu$ with $\lambda \neq \mu$. Let $L,M$ be the eigenspaces associated with eigenvalues $\lambda,\mu$ respectively. (That is, $L$ is the set of all eigenvectors with eigenvalue $\lambda$; $M$ is the set of all eigenvectors with eigenvalue $\mu$.)

Let $\{u_1,u_2,\ldots,u_p\}$ be a basis for $L$ and let $\{v_1,v_2,\ldots,v_q\}$ be a basis for $M$. Show that $\{u_1,u_2,\ldots,u_p,v_1,v_2,\ldots,v_q\}$ is a linearly independent set of $p+q$ vectors. (Hint: try $p=1$ and $q=1$ to start). (Comment: You could explore the case if there were three different eigenvalues and three bases for the eigenspaces).

Solution:

Assume that $\{u_1,u_2,\ldots,u_p,v_1,v_2,\ldots,v_q\}$ are linearly dependent, namely we can find $a_1,a_2,\ldots,a_{p+q}$ with $a_1u_1+a_2u_2+\ldots+a_pu_p+a_{p+1}v_1+a_{p+2}v_2+\ldots+a_{p+q}v_q=0$. Let $x=a_1u_1+a_2u_2+\ldots+a_pu_p$ and $y=a_{p+1}v_1+a_{p+2}v_2+\ldots+a_{p+q}v_q$ so that $Ax=\lambda x$ and $Ay=\mu y$ with $x+y=0$.

Case 1. $x \neq 0$ and so $y \neq 0$.

We have $0=A0=A(x+y)=\lambda x+\mu y$. But now one of $\lambda,\mu \neq 0$ (they are not equal), say $\lambda \neq 0$ and so $x=(-\mu/\lambda)y$. But then $\lambda x=Ax=(-\mu/\lambda)Ay=(\mu^2/\lambda)y$ and then this forces $\lambda(-\mu/\lambda)=(-\mu^2/\lambda)$ which has $\lambda=\mu$ as the only conclusion, a contradiction.

Case 2. $x=0$ and so $y=0$.

With $x=0$, we have $0=a_1u_1+a_2u_2+\ldots+a_pu_p$ and so by the linear independence of $u_1,u_2,\ldots,u_p$ means $a_1=a_2=\cdots=a_p=0$ and similarly with $y=0$, we have $0=a_{p+1}v_1+a_{p+2}v_2+\ldots+a_{p+q}v_q$ and the linear independence of $v_1,v_2,\ldots,v_q$ forces $a_{p+1}=a_{p+2}=\cdots=a_{p+q}=0$.

Thus we have shown that Case 1 cannot occur and Case 2 yields linear independence.

What if there were three different eigenvalues and three bases for the eigenspaces? It suffices to consider $u,v,w$ eigenvectors with $Au=\lambda u$, $Av=\mu v$ and $Aw=\rho w$ where $\lambda \neq \mu \neq \rho$.

Assume $au+bv+cw=0$. Then $A(au+bv+cw)=A0=0$. But $A(au+bv+cw)=a\lambda u+b\mu v+c\rho w=0$. Now $a\lambda u+b\mu v+c\rho w=0$ yields $a\lambda u+b\mu v+c\rho w=0$ and so we deduce $b(\mu-\lambda)v+c(\rho-\lambda)w=0$. But we already have $v,w$ are linearly independent and so $b(\mu-\lambda)=0$ and $c(\rho-\lambda)=0$. Given that $\lambda \neq \mu \neq \rho$, we deduce that $b=c=0$. From this we deduce that $a=0$ and hence $u,v,w$ are linearly independent. This is the important part in showing that if we have three different eigenvalues and three bases for the eigenspaces then the union of the bases are linearly independent. You can probably see this also works for 4 or more different eigenvalues.

6. Let $R^{n \times n}$ denote the vector space of all $n \times n$ matrices (over $R$). Consider following transformation $f: R^{n \times n} \rightarrow R^{n \times n}$

$$f(A) = A^T.$$ 

Show that this is a linear transformation.

We say that a matrix $A$ is symmetric if $A^T = A$ and we say that a matrix $A$ is skew-symmetric if $A^T = -A$.

a) Warmup question: Give a basis for $R^{n \times n}$. How many elements are in your basis?

b) What is the dimension of the eigenspace of eigenvalue 1 for $f$? Explain.

c) What is the dimension of the eigenspace of eigenvalue -1 for $f$? Explain.
d) Now use the previous question (and other facts) to show that any $A \in \mathbb{R}^{n \times n}$ is a linear combination of a symmetric matrix and a skew-symmetric matrix (you could show this directly of course but I’m asking you to use linear independence/dimension arguments).

**Solution:**

a) All matrices of the form $e_i e_j^T$, $1 \leq i, j \leq n$. There are $n^2$.

b) This eigenspace is the set of matrices satisfying $A^T = A$. A basis is the set of matrices of the form $e_i e_j^T + e_j e_i^T$, $1 \leq i < j \leq n$ unioned with the set of matrices of the form $e_i e_i^T$, $1 \leq i \leq n$. The dimension is the number of basis elements: $n(n + 1)/2$.

c) This eigenspace is the set of matrices satisfying $A^T = -A$. A basis is the set of matrices of the form $e_i e_j^T - e_j e_i^T$, $1 \leq i < j \leq n$. The dimension is $n(n - 1)/2$.

d) Using the result of Question 5), we find that the union of bases from Part b) and Part c) is a linearly independent set. There are $n(n + 1)/2 + n(n - 1)/2 = n^2$ elements in this set. Thus, the union must be a basis for $\mathbb{R}^{n \times n}$ (since $\mathbb{R}^{n \times n}$ has dimension $n^2$). This implies that any matrix can be written as a linear combination of elements from the two bases. We can decompose that linear combination into the part from eigenspace of eigenvalue 1, and the part from eigenspace of eigenvalue -1, i.e., a symmetric matrix plus a skew-symmetric matrix.