

Section 8.3

Sums and Moment Generating Function

Functions of independent r.v.s are independent

Fact: Let X, Y be independent. Then $f(X)$ and $g(Y)$ are independent for functions f, g .

- Provided everything is well defined.

Recall: Moment generating function (MGF)

$$M_X(t) := \mathbb{E}e^{tX}$$

Q) Let X, Y be independent. What is $M_{X+Y}(t)$?

$$M_{X+Y}(t) = \mathbb{E}[e^{t(X+Y)}] = \mathbb{E}[\underbrace{e^{tX}}_{\text{indep}} \cdot \underbrace{e^{tY}}_{\text{indep}}] = \mathbb{E}[e^{tX}] \cdot \mathbb{E}[e^{tY}] = M_X(t) \cdot M_Y(t)$$

Fact: Let X, Y be independent. Then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$.

- In words: “Mgf of sum is product of individual mgfs.”
(Much simpler than pmf/pdf of sum, which involves convolution!)
- Since mgf characterizes distribution, this gives an easy way to characterize distribution of a sum.

Ex) Let $X_1, X_2, \dots, X_n \sim N(0,1)$ be independent. What is the distribution of $S_n = X_1 + X_2 + \dots + X_n$?

Recall: For $X \sim N(0, \sigma^2)$, $M_X(t) = e^{\frac{t^2 \sigma^2}{2}}$.

$$\begin{aligned} M_{S_n}(t) &= M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t) = (M_{X_1}(t))^n \\ &= \left(e^{t^2/2} \right)^n = e^{\frac{t^2 \cdot n}{2}} \end{aligned}$$

$$\Rightarrow \boxed{S_n \sim N(0, n)}$$

Q) Let $X \sim \text{Bin}(n, p)$. What is the mgf of X ?

$X \stackrel{\text{dist}}{=} X_1 + X_2 + \dots + X_n$ $X_i \sim \text{Bern}(p)$ are indep

$$\begin{aligned} M_X(t) &= M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t) \\ &= (M_{X_1}(t))^n = \boxed{(1-p + pe^t)^n} \end{aligned}$$

