

7.1

Sums of independent r.v.s

(Note: 7.2 not part of class material)

Q) What is the distribution of $X + Y$?

• What is pdf or pmf of $X+Y$?

• We learn new operator: convolution

Ex) Pmf of X, Y given in table.

$$P(X + Y = 0) = \underline{P(0, 0) = \frac{1}{16}}$$

$$P(X + Y = 1) = \underline{P(1, 0) + P(0, 1)} \\ = \underline{\frac{1}{16} + \frac{1}{16}}$$

$$P(X + Y = 2) = \underline{P(2, 0) + P(1, 1) + P(0, 2) = \frac{1}{8} + \frac{1}{16} + \frac{1}{8}}$$

$$P(X + Y = 3) = \underline{\frac{1}{8} + \frac{1}{8}}$$

$$P(X + Y = 4) = \underline{\frac{1}{4}}$$

	X		
	0	1	2
Y	0	$\frac{1}{16}$	$\frac{1}{16}$
	1	$\frac{1}{16}$	$\frac{1}{8}$
	2	$\frac{1}{8}$	$\frac{1}{4}$

Generalizing, discrete case

If X, Y have joint pmf p and take values in $\{0, 1, 2, \dots\}$,

then $P(X + Y = n) = p(0, n) + \underline{p(1, n-1) + p(2, n-2) + \dots + p(n, 0)}$

More generally, if X, Y have joint pmf p . Then

$$P(X + Y = n) = \sum_k \underline{p(k, n-k)}$$

summed over k
such that (X, Y)
can take values
 $(k, n-k)$.

If X, Y are independent, then this becomes:

$$P(X + Y = n) = \sum_k \underline{P_X(k) \cdot P_Y(n-k)}$$

Pmf of sum of indep. r.v.s is convolution of individual pmfs

Fact: Let X, Y be independent. Then $X + Y$ satisfies:

Discrete case:

$$p_{X+Y}(n) = \sum_k p_X(k) p_Y(n-k) =: p_X * p_Y(n) = p_Y * p_X(n)$$

Continuous case:

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx =: f_X * f_Y(z) = f_Y * f_X(z)$$

- $p_X * p_Y$ is the *convolution* of p_X and p_Y . In words, “the pmf for $X + Y$ is the convolution of the pmf for X with the pmf for Y ”.
- $f_X * f_Y$ is the convolution of f_X and f_Y

Examples

(Using convolution to find pmf or pdf of $X + Y$)

Ex) $X \sim \text{Poisson}(\lambda), Y \sim \text{Poisson}(\alpha), X, Y$ are independent. What is pmf of $X + Y$?

$$p_{X+Y}(n) = p_X * p_Y(n) = \sum_{k=0}^n P_X(k) \cdot P_Y(n-k)$$

$$= \sum_{k=0}^n \frac{e^{-\lambda} \lambda^k}{k!} e^{-\alpha} \frac{\alpha^{n-k}}{(n-k)!}$$

$$= (\lambda + \alpha)^n \underbrace{e^{-\lambda} \cdot e^{-\alpha}}_{n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \cdot \underbrace{\frac{\lambda^k}{(\lambda + \alpha)^k}}_{p^k} \cdot \underbrace{\frac{\alpha^{n-k}}{(\lambda + \alpha)^{n-k}}}_{(1-p)^{n-k}}$$

$$= \boxed{\frac{(\lambda + \alpha)^n e^{-(\lambda + \alpha)}}{n!}}$$

$$P(\text{Bin}(n, p) \in \{0, 1, \dots, n\}) = 1$$

for $n \in \{0, 1, \dots\}$.

I.e. $X + Y \sim \text{Poisson}(\lambda + \alpha)$

$$k \in \{0, 1, 2, \dots\}$$

$$n-k \in \{0, 1, 2, \dots\}$$

$$\boxed{P_X(x) = \frac{e^{-\lambda} \lambda^k}{k!} \quad k \in \{0, 1, 2, \dots\}}$$

Ex) $X, Y \sim N(0,1)$ are independent. What is pdf of $X + Y$?

$$f_Z(z) = f_X * f_Y(z) \quad f_X = \phi = f_Y$$

$$= \int_{-\infty}^{\infty} \phi(x) \phi(z-x) dx$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$= \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-x)^2} dx$$

$$= \frac{1}{2\pi} \cdot \int \exp\left(-\frac{1}{2} \underbrace{\left[x^2 + (z-x)^2\right]}_a\right) dx$$

$$a = x^2 + z^2 - 2zx + x^2$$

$$= 2x^2 - 2zx + z^2 = 2\left(x - \frac{z}{2}\right)^2 - \frac{z^2}{4} + z^2$$

$$= 2 \left(x - \frac{z}{2} \right)^2 + \frac{z^2}{2}$$

$$f_z(z) = \frac{1}{2\pi} \int \exp \left(-\frac{1}{2} \left[2 \left(x - \frac{z}{2} \right)^2 + \frac{z^2}{2} \right] \right) dx$$

$$= \frac{1}{2\pi} \int e^{-\frac{1}{2} \frac{(x - \frac{z}{2})^2}{1/2}} \cdot e^{-\frac{1}{2} \frac{z^2}{2}} dx$$

$$= e^{-\frac{1}{2} \frac{z^2}{2}} \frac{1}{\sqrt{2\pi \cdot 2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \cdot \frac{1}{2}} e^{-\frac{1}{2} \frac{(x - \frac{z}{2})^2}{1/2}} dx$$

$N(\frac{z}{2}, 1/2)$

$$P(N(\frac{z}{2}, 1/2) \in (-\infty, \infty)) = 1$$

$$= \frac{1}{\sqrt{2\pi \cdot 2}} \cdot e^{-\frac{1}{2} \frac{z^2}{2}}$$

$z \sim N(0, 2)$

Ex) $X, Y \sim \text{Unif}[0,1]$. What is the pdf of $X + Y$?

X, Y are indep

$$f_Z(z) = f_X * f_Y(z)$$

$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x) dx$$

$$= \int \mathbb{1}_{[x \in [0,1]]} \cdot \mathbb{1}_{[x \in [z-1, z]]} dx$$

$$= \int \mathbb{1}_{[x \in [0,1] \cap [z-1, z]]} dx$$

$$= \text{length}([0,1] \cap [z-1, z])$$

z

$$f_X(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{else} \end{cases} = \mathbb{1}_{[x \in [0,1]]}$$

$$f_X(z-x) = \mathbb{1}_{[z-x \in [0,1]]}$$

$-x \in [-z, 1-z]$
 $x \in [z-1, z]$

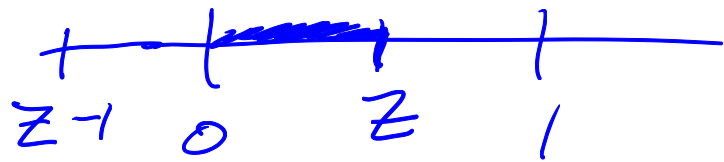
Case 1 $z < 0$

$$[0,1] \cap [z-1, z] = \emptyset, \quad \text{length} = 0$$

Case 2 $z \in [0, 1]$

$$[0, 1] \cap [z-1, z] \\ = [0, z]$$

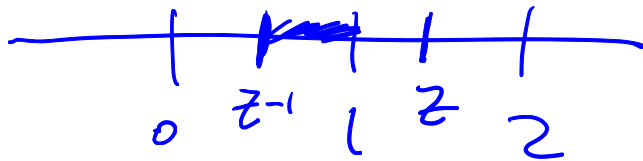
$$\text{length} = z$$



Case 3: $z \in [1, 2]$

$$[0, 1] \cap [z-1, z] \\ = [z-1, 1]$$

$$\text{length} = 1 - (z-1) = 2 - z$$



Case 4: $z > 2$
 $\text{length} = 0$

$$f_z(z) = \begin{cases} z & z \in [0, 1] \\ 2-z & z \in (1, 2] \\ 0 & \text{else} \end{cases}$$

