

• Midterm tomorrow

Section 6.3

Joint distributions and independence

Jointly discrete and independent

Fact: Let X, Y have joint pmf p and marginal pmfs p_X, p_Y . Then X and Y are independent if and only if

$$\underbrace{p(k, l)}_{P(X=k, Y=l)} = \frac{\underbrace{p_X(k) \cdot p_Y(l)}_{P(X=k) \cdot P(Y=l)}}{1}$$

for all possible values of k, l .

Ex) X, Y are the results of two die rolls.

Marginal pmfs: $p_X(k) = \frac{1}{6}$, $p_Y(l) = \frac{1}{6}$ } $k, l \in \{1, 2, \dots, 6\}$

Joint pmf: $p(k, l) = \frac{p_X(k) \cdot p_Y(l)}{1} = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

Ex) Roll two dice. X is first die roll, S is the sum. Are X, S independent?

They are dependent. Indeed,

$$P(X=1, S=12) = 0$$

$$\underbrace{P(X=1)}_{1/6} \cdot \underbrace{P(S=12)}_{1/36} \neq 0$$



Ex) Now, X is the first die roll and

$Z = 2^{\text{nd}}$ die roll

$$Y = \begin{cases} 1, & \text{if sum is 7} \\ 0, & \text{else.} \end{cases}$$

Are X, Y independent?

$$P(X=k, Y=l) \stackrel{?}{=} P(X=k) \cdot P(Y=l) \quad \text{for all } k \in \{1, 2, \dots, 6\}, l \in \{0, 1\}$$

$$l=1, k \in \{1, 2, \dots, 6\}$$

$$\bullet P(X=k, Y=1) = P(X=k, Z=7-k) = P(X=k) \cdot P(Z=7-k) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

$$\bullet P(X=k) \cdot P(Y=1) = \frac{1}{6} \cdot \frac{6}{36} = \frac{1}{36}$$

$$\Rightarrow \{X=k\} \text{ \& \& } \{Y=1\} \text{ are indep.}$$

$$\Rightarrow \{X=k\} \text{ \& \& } \{Y=0\} \text{ are indep.}$$

$$\text{i.e. } P(X=k, Y=l) = P(X=k) \cdot P(Y=l) \quad \text{for all } k \in \{1, \dots, 6\}$$

Recall If A, B
are indep, then
so are A & B^c

$\ell \in \mathbb{R}$ so X, Y are indep.

Jointly continuous and independent

Fact: For jointly continuous, independent r.v.s, X, Y with joint density f and marginal densities f_X and f_Y ,

$$f(x, y) = \underline{f_X(x) \cdot f_Y(y)} \quad \text{for all } x, y \in \mathbb{R}.$$

Ex) $X \sim \text{Exp}(\lambda), Y \sim \text{Exp}(\alpha)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

$$f(x, y) = \begin{cases} \lambda \cdot \alpha e^{-\lambda x} \cdot e^{-\alpha y} & x, y \geq 0 \\ 0 & \text{else} \end{cases}$$

$$f_Y(y) = \begin{cases} \alpha e^{-\alpha y} & y \geq 0 \\ 0 & \text{else} \end{cases}$$

Nuanced version of cont. and independent

Fact: Let X, Y be r.v.s on the same sample space with marginal densities f_X and f_Y .

- If X, Y have joint density function

$$f(x, y) = \underline{f_X(x) \cdot f_Y(y)} \quad \text{for all } x, y \in \mathbb{R}$$

then X, Y are independent.

- Conversely, if X, Y are independent, then they are jointly continuous with joint density function

$$f(x, y) = \underline{f_X(x) \cdot f_Y(y)} \quad \text{for all } x, y \in \mathbb{R}$$

Ex) Multivariate Gaussian

Let $X, Y \sim N(0,1)$ be independent standard normal r.v.s. What is their joint density function?

$$f_X(x) = \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad f_Y(y) = \phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

Joint density: $f(x, y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot e^{-\frac{1}{2}y^2}$
 $= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \quad x, y \in \mathbb{R}$

