

Section 6.2

Jointly continuous random variables

Def: R.v.s X, Y are called *jointly continuous* if there exists a *joint density function* $f: \mathbb{R}^2 \rightarrow \mathbb{R}^+$ such that for any subset $B \subset \mathbb{R}^2$,

$$P((X, Y) \in B) = \iint_B f(x, y) dx dy$$

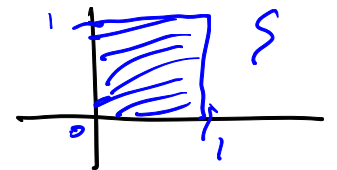
- $f(x, y) \geq \underline{0}$

- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) = \underline{1}$

Ex) Let (X, Y) be uniformly chosen from the square, $S = [0, 1] \times [0, 1]$.

Then,

$$f(x, y) = \begin{cases} 1 & (x, y) \in S \\ 0 & (x, y) \notin S \end{cases}$$



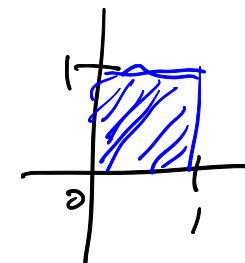
Expected value

Fact: Let $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ and let X, Y be r.v.s with joint density function f .
Then

$$\mathbb{E}g(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy$$

Ex) Let (X, Y) be uniformly chosen from S . What is $\mathbb{E}[X + Y]$?

$$\begin{aligned}\mathbb{E}[X+Y] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+y) \cdot f(x,y) dx dy = \int_0^1 \int_0^1 (x+y) \cdot 1 dx dy \\ &= \int_0^1 \left. \frac{x^2}{2} + xy \right|_0^1 dy = \int_0^1 \left(\frac{1}{2} + y \right) dy = \left. \frac{1}{2}y + \frac{y^2}{2} \right|_0^1 \\ &= \frac{1}{2} + \frac{1}{2} = \boxed{1}\end{aligned}$$



$$f(x,y) = \begin{cases} 1 & (x,y) \in S = [0,1] \times [0,1] \\ 0 & \text{else} \end{cases}$$

Now, what is $\mathbb{E}|X - Y|$?

$$|x-y| = \begin{cases} x-y & x \geq y \\ y-x & x < y \end{cases}$$

$$\mathbb{E}|X-Y| = \int_0^1 \int_0^1 |x-y| dx dy = \int_0^1 \int_0^y \underbrace{y-x}_a dx dy + \int_0^1 \int_y^1 \underbrace{x-y}_b dx dy$$

$$a = \int_0^1 y \cdot x - \frac{x^2}{2} \Big|_0^y dy$$

$$= \int_0^1 y^2 - \frac{y^2}{2} dy = \int_0^1 \frac{y^2}{2} dy = \left. \frac{y^3}{6} \right|_0^1 = \frac{1}{6} = b \xrightarrow{\text{exercise}} \Rightarrow \mathbb{E}|X-Y| = \frac{1}{6} + \frac{1}{6} = \boxed{\frac{1}{3}}$$

Jointly continuous vs continuous marginals

Q) If X is continuous and Y is continuous, must (X, Y) be jointly continuous?

No! Counterexample: $X \sim \text{Unif}[0,1]$, $Y = X$

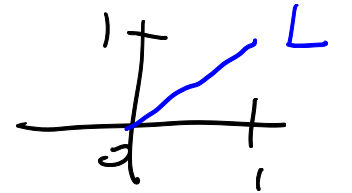
Suppose, for contradiction, (X, Y) are jointly cont. Then

they have some pdf f . Let $L = \{(x, y) : x = y, x, y \in [0, 1]\}$

$$P((X, Y) \in L) = \int_L \int f(x, y) dx dy = 0$$

← 1-dim line!

However, $X = Y$, so $P((X, Y) \in L) = P(X = Y) = 1 \Rightarrow \Leftarrow$



Q) If (X, Y) are jointly continuous, must X and Y each be continuous?

Yes!

Marginals

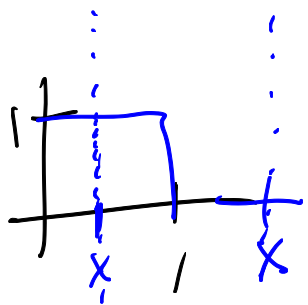
Fact: Let X, Y have joint density function f . Then X is continuous with pdf f_X satisfying

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

Ex) (X, Y) are chosen uniformly from $S = [0, 1] \times [0, 1]$

$$f(x, y) = \begin{cases} 1 & (x, y) \in S \\ 0 & (x, y) \notin S \end{cases} \quad \text{For } x \in [0, 1],$$
$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 1 dy = 1$$

$$\text{For } x \notin [0, 1], \quad f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_{-\infty}^{\infty} 0 dy = 0 \Rightarrow f_X(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{else} \end{cases}$$
$$\Rightarrow X \sim U_n([0, 1])$$



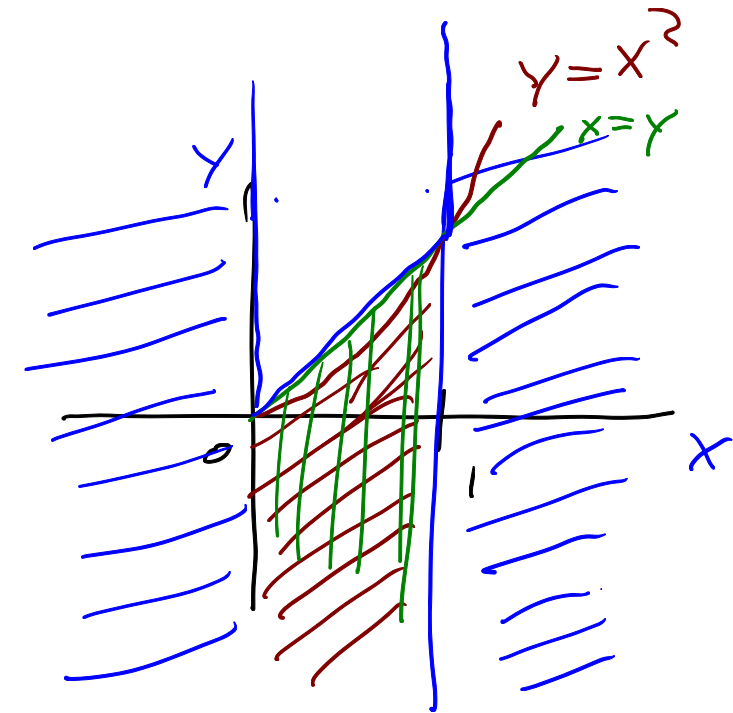
Technical challenge: Finding bounds of integration

Ex) Let (X, Y) have joint pdf

$$f(x, y) = \begin{cases} 2xe^{x^2-y}, & \text{if } \overset{1}{0 < x < 1} \text{ and } \overset{2}{y > x^2} \\ 0, & \text{else.} \end{cases}$$

Q) What is $P(X < Y)$?

$$\begin{aligned} P(X < Y) &= \int \int_{\substack{X < Y \\ 3}} f(x, y) dx dy \\ &= \int_0^1 \int_x^\infty 2xe^{x^2-y} dy dx \end{aligned}$$



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Uniform r.v.s in higher dimensions

Def: Let D be a subset of $\mathbb{R}^2$ with $\text{area}(D) < \infty$. The random point (X, Y) is uniformly distributed on D if its joint density function is

$$f(x, y) = \begin{cases} \frac{1}{\text{area}(D)} & (x, y) \in D \\ 0 & (x, y) \notin D \end{cases}$$

- Notation: $(X, Y) \sim \text{Unif}(D)$
- In three dimensions, area is replaced by volume.
- In n dimensions, it is replaced by *hypervolume* (volume in > 3 dimensions)

Ex) Uniform on ball

$$f(x,y) = \begin{cases} 1/\pi & (x,y) \in B \\ 0 & (x,y) \notin B \end{cases}$$

Let $(X,Y) \sim \text{Unif}(B)$ where $B = \{(x,y): x^2 + y^2 \leq 1\}$.

Q) What is the marginal of X ?

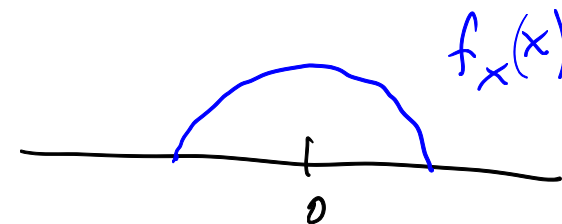
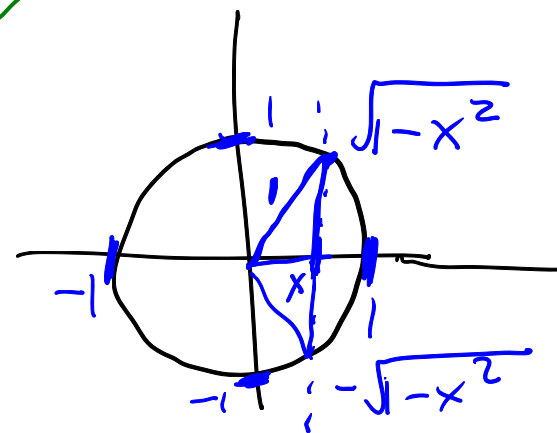
$f_X(x) = ?$ For $x \notin [-1,1]$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y) dy = \int_{-\infty}^{\infty} 0 dy = 0$$

For $x \in [-1,1]$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y) dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2\sqrt{1-x^2}}{\pi}$$

$$\Rightarrow f_X(x) = \begin{cases} \frac{2\sqrt{1-x^2}}{\pi} & x \in [-1,1] \\ 0 & \text{else} \end{cases}$$



Ex) Uniform on high-dimensional ball

Let $(X_1, X_2, \dots, X_n) \sim \text{Unif}(\sqrt{n}B_n)$ where

$$\sqrt{n} B_n = \{(x_1, x_2, \dots, x_n) : x_1^2 + \dots + x_n^2 \leq n\}$$

What is the marginal of X_1 ?

Set $V_n = \text{Vol}(B_n)$, Note: for $r > 0$, $\text{Vol}(r B_n) = r^n \text{Vol}(B_n)$

$$\text{Vol}(\sqrt{n} B_n) = \sqrt{n}^n V_n = n^{n/2} V_n$$

$$f(x_1, x_2, \dots, x_n) = \begin{cases} \frac{1}{n^{n/2} V_n} & (x_1, x_2, \dots, x_n) \in \sqrt{n} B_n \\ 0 & \text{else} \end{cases}$$

$$f_{x_1}(x_1) = \frac{1}{n^{n/2} V_n} \cdot \text{Vol}(\sqrt{n-x_1^2} B_{n-1})$$

$$= \frac{1}{n^{n/2} V_n} \cdot \sqrt{n-x_1^2}^{n-1} \cdot V_{n-1}$$

$$= \frac{V_{n-1}}{V_n} \frac{(n-x_1^2)^{\frac{n-1}{2}}}{n^{n/2}} = \frac{V_{n-1}}{V_n} \cdot \frac{(n-x_1^2)^{\frac{n}{2}}}{n^{n/2}} \frac{1}{\sqrt{n-x_1^2}} = \frac{\sqrt{n}}{\sqrt{n}}$$

$$= \underbrace{\frac{V_{n-1}}{V_n} \frac{\sqrt{n}}{\sqrt{n}}}_a \cdot \underbrace{\frac{1}{\sqrt{n-x_1^2}}}_b \cdot \underbrace{\left(1 - \frac{x_1^2}{n}\right)^{n/2}}_c$$

$$b = \frac{1}{\sqrt{1 - \frac{x_1^2}{n}}} \xrightarrow{n \rightarrow \infty} 1, \quad c = \left(1 - \frac{x_1^2}{n}\right)^{n/2} \xrightarrow{n \rightarrow \infty} e^{-\frac{x_1^2}{2}}$$

$$a \xrightarrow{n \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \Rightarrow f_{x_1}(x_1) \xrightarrow{n \rightarrow \infty} \boxed{\frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}}}$$

Challenge exercise

