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Joint distribution of random variables

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Random vector

$$(X_1, X_2, \dots, X_n)$$

Q) What is the *joint distribution* of all r.v.s?

Q) Given the joint distribution, how to do you find *marginal distributions*, i.e., distributions for each individual r.v. X_i ?

Q) How to compute $\mathbb{E}g(X_1, X_2, \dots, X_n)$?

Simplification: Consider two r.v.s (X, Y)

- All ideas will generalize naturally to more r.v.s.
- See book (or ask me questions) regarding the case or more r.v.s

6.1

Joint^t distribution of discrete random variables

Def: Let X, Y be discrete r.v.s defined on the same sample space. Their joint probability mass function (joint pmf) is defined by

$$p(k, l) = P(X = k, Y = l)$$

for all values k and l that X and Y can take.

Ex) Roll two dice, X = first die roll, Y = second die roll.

$$p(1,1) = p(1,2) = \dots = p(5,6) = p(6,6) = \underline{\quad \frac{1}{36} \quad}$$

Note: In this case X, Y are independent, more on that later.

Ex) Flip coin 3 times. Sample space: $\Omega = \{HHH, HHT, \dots, TTT\}$.

- X = number of heads
- $Y = \begin{cases} 1 & \text{if you get at least 2 heads in a row} \\ 0 & \text{otherwise} \end{cases}$

		X			
		0	1	2	3
Y	0	TTT	HTT, THT, TTH	HTH	{ }
	1	{ }	{ }	HHT, TTH	HHH

samples corr.
w/ $X=i, Y=j$

Joint pmf table: Entry $(i, j) = P(Y = i, X = j)$

		X			
		0	1	2	3
Y	0	$p(0,0) = 1/8$	$3/8$	$1/8$	0
	1	0	0	$2/8$	$1/8$

What about the *marginal distributions*, i.e., how to find pmf of X and pmf of Y ?

		X				
		0	1	2	3	
Y	0	1/8	3/8	1/8	0	5/8
	1	0	0	2/8	1/8	3/8
		1/8	3/8	3/8	1/8	

$$P(X=0) = P(X=0, Y=0) + P(X=0, Y=1) = \frac{1}{8} + 0 = \frac{1}{8}$$

$$\begin{aligned}
 P(Y=0) &= P(Y=0, X=0) + P(Y=0, X=1) + P(Y=0, X=2) + P(Y=0, X=3) \\
 &= \frac{1}{8} + \frac{3}{8} + \frac{1}{8} + 0
 \end{aligned}$$

Marginal pmf

Def: Let $p(k, l)$ be the joint pmf of (X, Y) . Then the pmf of X is given by

$$p_X(k) = \sum_l p(k, l).$$

p_x is called the *marginal pmf* of X .

Similarly, $p_Y(l) = \underline{\sum_k p(k, l)}$

Expected value of $g(X, Y)$

Def: Let $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ and let X, Y be discrete r.v.s with joint pmf p .
Then

$$\mathbb{E}g(X, Y) = \sum_{k, l} g(k, l) p(k, l)$$

summed over
 k, l vals
 X, Y can take

Ex) Flip 3 coins, you win 1 dollar for each head, but only if there are at least two heads in a row.

		X			
		0	1	2	3
Y	0	1/8	3/8	1/8	0
	1	0	0	2/8	1/8

Dollars won = $X \cdot Y$

$$\begin{aligned}
 \mathbb{E} X \cdot Y &= 0 \cdot 0 \cdot \cancel{P(0,0)}^{1/8} + 1 \cdot 0 \cdot \cancel{P(1,0)}^{3/8} + 2 \cdot 0 \cdot \cancel{P(2,0)}^{1/8} + 3 \cdot 0 \cdot \cancel{P(3,0)}^0 \\
 &\quad + 0 \cdot 1 \cdot \cancel{P(0,1)}^0 + 1 \cdot 1 \cdot \cancel{P(1,1)}^0 + 2 \cdot 1 \cdot \cancel{P(2,1)}^{3/8} + 3 \cdot 1 \cdot \cancel{P(3,1)}^{1/8} \\
 &= \boxed{7/8}
 \end{aligned}$$

Prisoner's dilemma

Two bank robbers, Xavier, and Yolanda, are caught and questioned. They each have two options.

Option 1: Testify against accomplice

Option 0: Remain silent

Let X be Xavier's choice, Y be Yolanda's choice. Suppose that Xavier chooses Option 1 with probability p , and Yolanda chooses Option 1 with probability q (independently), so

$$\begin{aligned} p(1,1) &= pq, & p(1,0) &= \underline{p(1-q)}, \\ p(0,1) &= \underline{(1-p)q}, & p(0,0) &= \underline{(1-p)(1-q)}. \end{aligned}$$

Q: What is the expected total number of years in jail?

$T = \text{total yrs}$

$$ET = 2 p(0,0) + 5 p(0,1) + 5 p(1,0) + 6 \cdot p(1,1)$$

$$= 2 \cdot (1-p)(1-q) + 5 \cdot p(1-q) + 5 \cdot (1-p)q + 6 \cdot p \cdot q$$

$$\begin{aligned} \frac{d}{dp} ET &= -2(1-q) + 5(1-q) - 5q + 6q \\ &= 3(1-q) + q = 3 - 2q > 0 \end{aligned}$$

$\Rightarrow$ If p goes up, so does ET .

Years in jail:

	$X = 0$	$X = 1$
$Y = 0$	(1,1)	(0,5)
$Y = 1$	(5,0)	(3,3)

If $x=0, y=0$
each spends
1 yr in
jail

If $x=1,$
 $y=0,$
Xavier spends
0 yrs,
Yolanda 5

Q: What is the expected number of years in jail for Xavier?

$T_x = \text{yrs for Xavier}$

$$\mathbb{E} T_x = 1 p(0,0) + 0 p(1,0) + 5 p(0,1) + 3 p(1,1)$$

$$= (1-p)(1-q) + 5(1-p) \cdot q + 3pq$$

$$\frac{d}{dp} \mathbb{E} T_x = -(1-q) - 5q + 3q = -1 + q - 2q = -1 - q < 0$$

Years in jail:

	$X = 0$	$X = 1$
$Y = 0$	(1,1)	(0,5)
$Y = 1$	(5,0)	(3,3)

So if p goes up $\mathbb{E}T_x$ goes down.