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Joint distribution of random variables:
Summary and Example problems

Analogues between univariate and multivariate quantities

	1 r.v. X	2 r.v.s X, Y
Pmf:	$p(k) = P(X = k)$	$p(k, l) = P(X = k, Y = l)$
Pdf:	$f(x)$	$f(x, y)$
Expected value of a function:	$\mathbb{E}g(x) = \sum_k g(k)p(k)$ or $\mathbb{E}g(x) = \int g(x)f(x)dx$	$\mathbb{E}g(x, y) = \sum g(k, l)p(k, l)$ or $\mathbb{E}g(x, y) = \int \int g(x, y)f(x, y)dxdy$
Probabilities	$P(X \in B) = \sum_{k \in B} p(k)$ or $P(X \in B) = \int_B f(x)dx$	$P((X, Y) \in B) = \sum_{(k, l) \in B} p(k, l)$ or $P((X, Y) \in B) = \underbrace{\int \int}_B f(x, y)dxdy$

Main new ideas with multiples r.v.s

Marginals: Given the joint distribution of X, Y , find the distribution of just X (and of just Y).

$$p_X(k) = \sum_l p(k, l), \quad f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

Independence: $p(k, l) = \frac{p_X(k) \cdot p_Y(l)}{1}, \quad f(x, y) = \frac{f_X(x) \cdot f_Y(y)}{1}$

Uniform r.v. in high dimensions:

Ex) Let $(x, y, z) \sim \text{Unif}(B)$, $B = \{(x, y, z): x^2 + y^2 + z^2 \leq r^2\}$.

$$\text{Then } f(x, y, z) = \begin{cases} \frac{1}{\frac{4}{3}\pi r^3} & (x, y, z) \in B \\ 0 & (x, y, z) \notin B \end{cases}$$

Technical challenge: Double integrals
(bounds of integration)

Ex) Let $(X, Y) \sim \text{Unif}(S)$, $S = [0, 1] \times [0, 1]$.

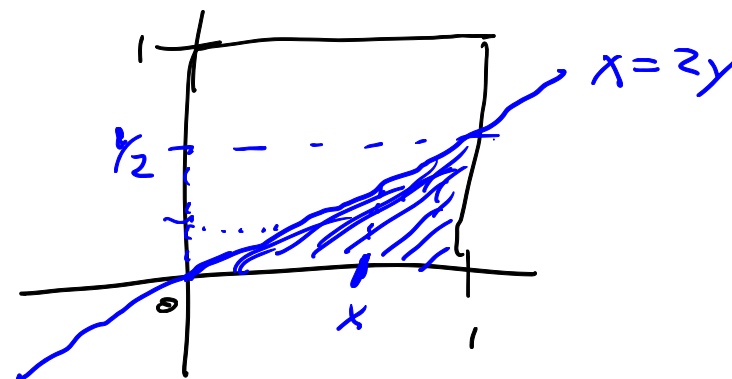
What is $P(X \geq 2Y)$?

$$P(X \geq 2Y) = P((X, Y) \in \underbrace{\{(x, y) : x \geq 2y\}}_B)$$

$$= \int_B \int f(x, y) dy dx$$

$$= \int_0^1 \int_0^{x/2} 1 dy dx$$

$$= \int_0^1 x/2 dx = \frac{x^2}{4} \Big|_0^1 = \boxed{\frac{1}{4}}$$



$$f(x, y) = \begin{cases} 1 & (x, y) \in S \\ 0 & \text{else} \end{cases}$$

Ex) Let $(X, Y) \sim \text{Unif}(S)$, $S = [0, 1] \times [0, 1]$.

$$f_{(X,Y)} = \begin{cases} 1 & (x,y) \in S \\ 0 & \text{else} \end{cases}$$

What are the marginal distributions of X and Y ?

$$f_X(x) = \int_{-\infty}^{\infty} f_{(X,Y)}(x,y) dy$$

For $x \notin [0, 1]$, $f_{(X,Y)} = 0$

so $f_X(x) = 0$

For $x \in [0, 1]$, $f_X(x) = \int_0^1 1 dx = 1$

$$\Rightarrow f_X(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{else} \end{cases} = \mathbb{1}_{[x \in [0, 1]]}$$

Similarly, $f_Y(y) = \mathbb{1}_{[y \in [0, 1]]}$

Are X and Y independent?

$$f_X(x) \cdot f_Y(y) \stackrel{?}{=} f_{(X,Y)} \quad \text{for all } x, y$$

$$\text{LHS} = \mathbb{1}_{[x \in [0, 1]]} \cdot \mathbb{1}_{[y \in [0, 1]]} = \mathbb{1}_{[(x,y) \in [0, 1] \times [0, 1]]} = f_{(X,Y)} \quad \checkmark$$

Yes, indep.

Ex) Suppose that X, Y have joint density function

$$f(x, y) = \begin{cases} \frac{12}{7}(xy + y^2), & 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\ 0, & \text{otherwise} . \end{cases}$$

- a) Check that f is a genuine density function.
- b) Find the marginal density function of Y .
- c) Calculate $P(X < Y)$.
- d) Calculate $\mathbb{E}X^2Y$.

$$f(x, y) = \begin{cases} \frac{12}{7}(xy + y^2), & 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

a) Check that f is a genuine density function.

Need 1) $f(x, y) \geq 0$ ✓, 2) $\iint f(x, y) = 1$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = \frac{12}{7} \int_0^1 \int_0^1 (xy + y^2) dx dy$$

$$= \frac{12}{7} \int_0^1 \left. \frac{x^2 y}{2} + xy^2 \right|_0^1 dy$$

$$= \frac{12}{7} \int_0^1 \left(\frac{y}{2} + y^2 \right) dy = \frac{12}{7} \cdot \left(\frac{y^2}{4} + \frac{y^3}{3} \right) \Big|_0^1$$

$$= \frac{12}{7} \left(\frac{1}{4} + \frac{1}{3} \right) = \boxed{1} \quad \checkmark$$

$$f(x, y) = \begin{cases} \frac{12}{7}(xy + y^2), & 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

b) Find the marginal density function of Y .

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx = \begin{cases} \frac{12}{7}(y/2 + y^2) & y \in [0, 1] \\ 0 & \text{else} \end{cases}$$

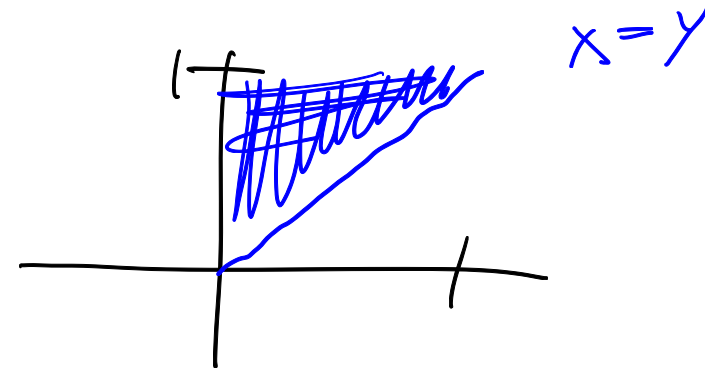
$$f(x, y) = \begin{cases} \frac{12}{7}(xy + y^2), & 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

c) Calculate $P(X < Y)$.

$$\int_0^1 \int_0^y \frac{12}{7}(xy + y^2) dx dy$$

...

$$\boxed{\frac{9}{14}}$$



Ans: $\frac{9}{14}$

$$f(x, y) = \begin{cases} \frac{12}{7}(xy + y^2), & 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

d) Calculate $\mathbb{E}X^2Y$.

$$\begin{aligned} \mathbb{E}[X^2Y] &= \iint x^2 y f(x, y) dx dy = \int_0^1 \int_0^1 x^2 y \cdot \frac{12}{7} (xy + y^2) dx dy \\ &\vdots \\ &= \boxed{\frac{2}{7}} \end{aligned}$$

Ans: $\frac{2}{7}$

Q) Let X, Y have pmf given by following table. Are X, Y independent?

		X			
		0	1	2	
Y	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{2}$ $P_Y(Y)$
	1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	
	2	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	
		$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$P_X(X)$

check that this holds for $x, y \in \{1, 2\}$

$$P_X(x) \cdot P_Y(y) = P_{X,Y}(x, y)$$