

Section 5.2

Distribution of a function of a random variable
(I.e., what is the dist. of $g(X)$?)

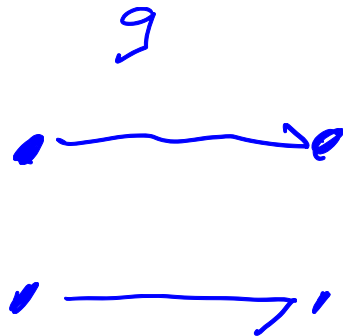
Ex) Let $X \sim \text{Bern}(p)$. What is the distribution of $Y := X^3 + 3$?

If $X = 0$, $Y = \underline{0^3 + 3 = 3}$, this occurs with probability $1-p$

If $X = 1$, $Y = \underline{1^3 + 3 = 4}$, this occurs with probability p

Thus, Y has pmf: $P_Y(3) = 1-p$, $P_Y(4) = p$

In above example, $Y = g(X) = X^3 + 3$, where g is a one-to-one function.



One-to-one function of discrete random variable:

Let X be a discrete random variable, g a one-to-one function, and $Y = g(X)$.

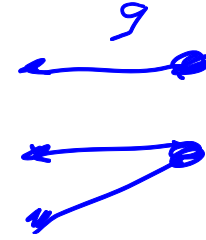
For each value l that Y can take, there is a value k of X such that $g(k) = l$. Then,
 $p_Y(l) = p_X(k) = p_X(g^{-1}(l))$.

Q) $X \sim \text{Bin}\left(2, \frac{1}{2}\right)$. What is the pmf of $\sqrt{X}$? $X \in \{0, 1, 2\}$

$P_X(0) = \frac{1}{4}, P_X(1) = \frac{1}{2}, P_X(2) = \frac{1}{4}$ $Y = \sqrt{X} \in \{0, 1, \sqrt{2}\}$

$P_Y(0) = P_X(0) = \frac{1}{4}, P_Y(1) = P_X(1) = \frac{1}{2}, P_Y(\sqrt{2}) = P_X(2) = \frac{1}{4}$
 $\downarrow$ since $g(0) = 0$

Discrete, but g is not one-to-one



Ex) Let X be uniformly chosen from $\{-1, 0, 1\}$. Let $Y = X^2$.

What values can Y take? 0, 1

$$p_Y(0) = \underline{p_X(0) = 1/3}$$

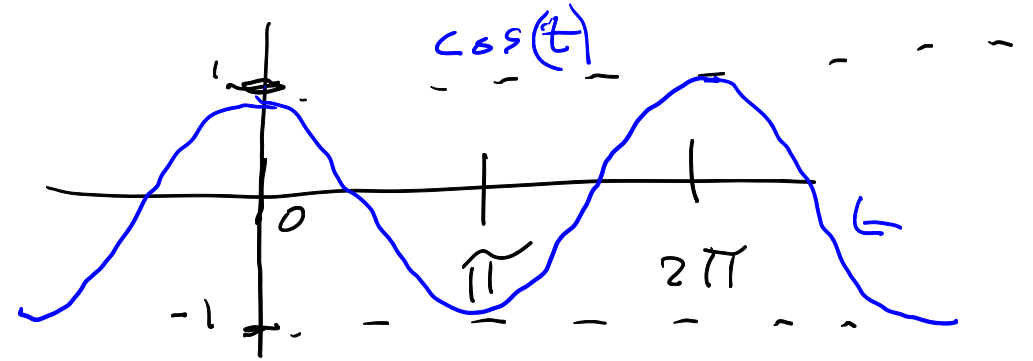
$$p_Y(1) = \underline{p_X(-1) + p_X(1) = 2/3}$$

General function of discrete random variable: Let X be a discrete r.v. with pmf p_X , let $Y = g(X)$. Then Y is discrete with pmf satisfying

$$p_Y(l) = \sum_{k: g(k)=l} p_X(k)$$

Q) Let $X \sim \text{Bin}\left(n, \frac{1}{2}\right)$, where n is odd. What is the pmf of $\cos(\pi X)$?

$$Y = \cos(\pi X) = \begin{cases} 1 & X \text{ even} \\ -1 & X \text{ odd} \end{cases}$$



$$Y \in \{1, -1\}$$

$$P_Y(1) = \sum_{k: \cos(\pi k) = 1} P_X(k) = \sum_{k \text{ even}} P_X(k) = P(X \text{ is even}) = \frac{1}{2}$$

$$P_Y(-1) = \sum_{k: \cos(\pi k) = -1} P_X(k) = \sum_{k \text{ odd}} P_X(k) = P(X \text{ is odd}) = \frac{1}{2}$$

exercise

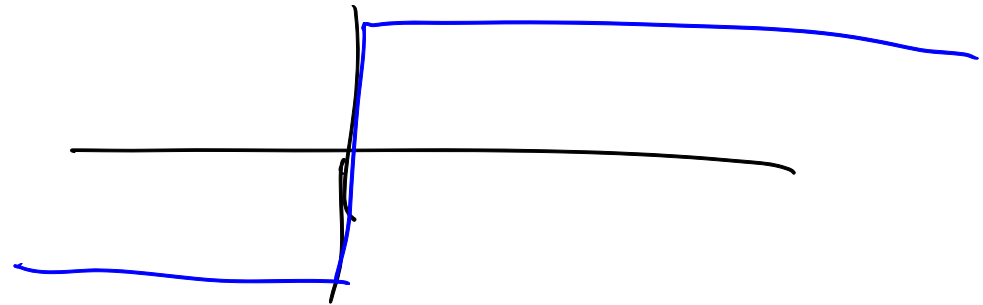
Continuous case

Above, $g(\text{"discrete r.v."}) = \text{"discrete r.v."}$

Q) Is $g(\text{"continuous r.v."}) = \text{"continuous r.v."}$?

Eg, $g(x) = 0$ is discrete since it only takes value 0.

$$\text{Eg. } g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$



Two methods for deriving dist of $g(X)$ in continuous case

- 1) Derive cdf
- 2) There is a formula for pdf of $g(X)$ similar to discrete case (which works if g is differentiable, and not “flat”).

Derive cdf

Ex) Let $U \sim \text{Unif}[0,1]$, $g(x) = -\frac{1}{\lambda} \ln(1-x)$. $\lambda > 0$

What is dist. of $Y := g(U)$?

Find cdf: $F_Y(s) = P(Y \leq s) = \underline{P(-\frac{1}{\lambda} \ln(1-x) \leq s)}$

$$= P(\ln(1-x) \geq -\lambda s) = P(1-x \geq e^{-\lambda s})$$

$$= P(x \leq 1 - e^{-\lambda s}) = 1 - e^{-\lambda s} \quad \text{for } s \geq 0$$

$$F_Y(s) = \begin{cases} 1 - e^{-\lambda s} & s \geq 0 \\ 0 & s < 0 \end{cases}$$

$= 0$ when $s < 0$

$$f_Y(s) = \frac{d}{ds} F_Y(s) = \begin{cases} \lambda e^{-\lambda s} & s \geq 0 \\ 0 & s < 0 \end{cases}$$
$$\Rightarrow \boxed{Y \sim \text{Exp}(\lambda)}$$

Ex) $Z \sim N(0,1)$, $Y := Z^2$.

Cdf: $F(s) = P(Y \leq s) = 0$ for $s < 0$ since $Y \geq 0$

$$\begin{aligned} \text{for } s \geq 0, \quad F(s) &= P(Y \leq s) = P(Z^2 \leq s) \\ &= P(-\sqrt{s} \leq Z \leq \sqrt{s}) \\ &= \Phi(\sqrt{s}) - \Phi(-\sqrt{s}) \end{aligned}$$

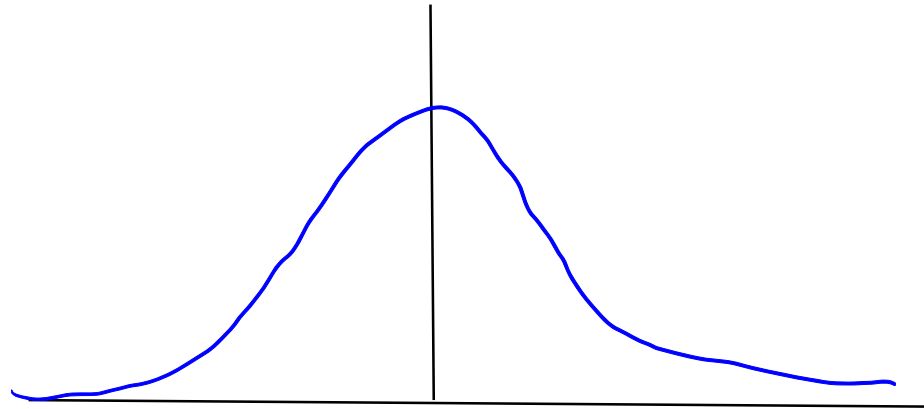
$$\begin{aligned} \text{Pdf: } f(s) &= \frac{d}{ds} F(s) = \frac{d}{ds} (\Phi(\sqrt{s}) - \Phi(-\sqrt{s})) \\ &= \frac{1}{2\sqrt{s}} \phi(\sqrt{s}) - \frac{1}{-2\sqrt{s}} \phi(-\sqrt{s}) \end{aligned}$$

$$\begin{aligned} f(s) &= 0 \quad \text{for } s < 0 \\ &= \frac{2}{2\sqrt{s}} \phi(\sqrt{s}) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{s}{2}} \quad \text{for } s \geq 0 \end{aligned}$$

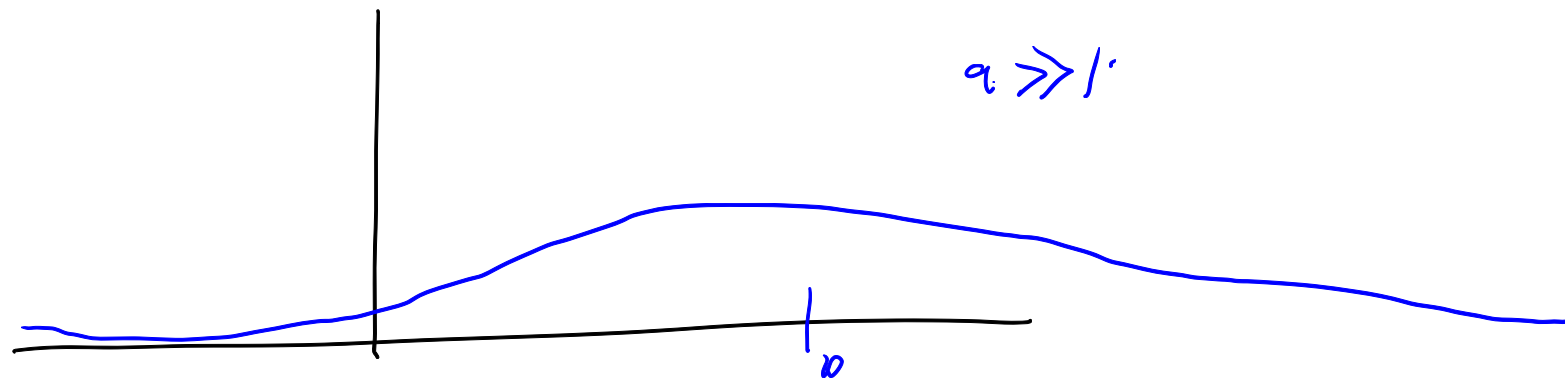
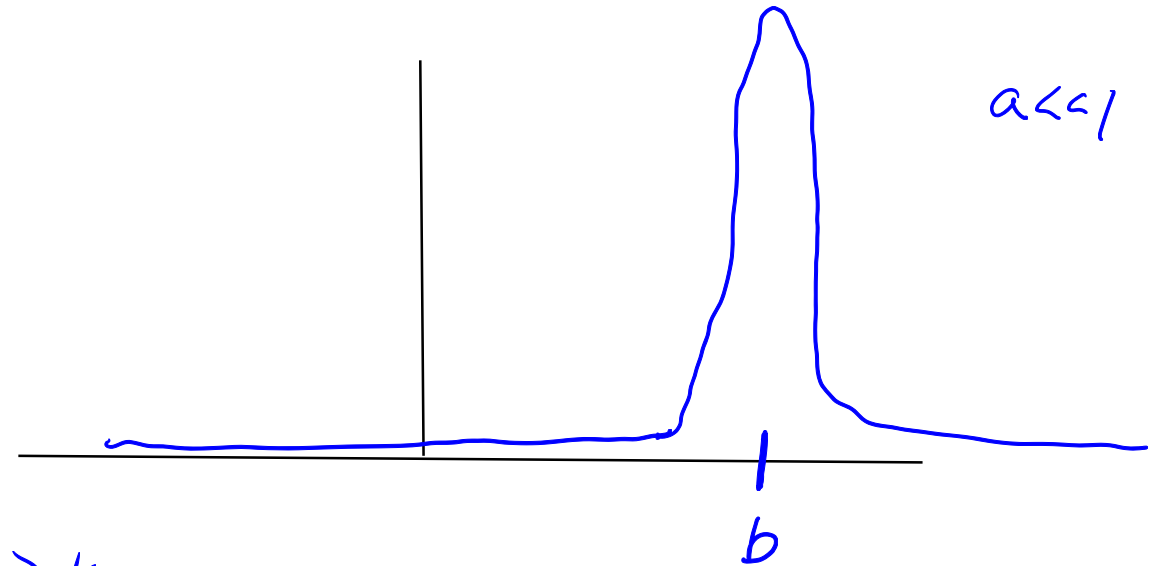
Pdf of affine transformation:

$$Y = g(X) = aX + b$$

Ex) Pdf of $X \sim N(0,1)$



Pdf of $Y \sim aX + b$



Affine transformation: $Y = g(X) = aX + b$

Cdf of Y :

$$F_Y(s) = \underline{P(aX + b \leq s)} = P(aX \leq s - b) -$$

Case 1: $a > 0$

$$F_Y(s) = P\left(X \leq \frac{s-b}{a}\right) = F_X\left(\frac{s-b}{a}\right)$$

$$\begin{aligned} f_Y(s) &= \frac{d}{ds} F_Y(s) = \frac{d}{ds} F_X\left(\frac{s-b}{a}\right) \\ &= \frac{1}{a} f_X\left(\frac{s-b}{a}\right) \end{aligned}$$

Case 2: $a < 0$

$$F_Y(s) = P\left(X \geq \frac{s-b}{a}\right)$$

$$= 1 - F_X\left(\frac{s-b}{a}\right)$$

$$\begin{aligned} f_Y(s) &= \frac{d}{ds} \left(1 - F_X\left(\frac{s-b}{a}\right)\right) \\ &= -\frac{1}{a} f_X\left(\frac{s-b}{a}\right) \end{aligned}$$

$$\Rightarrow \boxed{f_Y(s) = \frac{1}{|a|} f_X\left(\frac{s-b}{a}\right)}$$

Fact: Let X have pdf f_X and let $Y = aX + b$. Then

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y - b}{a}\right)$$

Ex) $X \sim N(0,1)$. $Y = aX + b$.

Pdf of Y : $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \frac{1}{\sqrt{2\pi}|a|} e^{-\frac{1}{2}\left(\frac{y-b}{a}\right)^2}$

i.e., $Y \sim N(b, a^2)$

One-to-one function of continuous random variable: Let X have density function f_X , and g be one-to-one, differentiable, with derivative equalling 0 at only finitely many points. Let $Y = g(X)$. Then

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}.$$

For points y where $g^{-1}(y)$ exists and $g'(g^{-1}(y)) \neq 0$. For other points $f_Y(y) = 0$.

Formula for pdf when g is differentiable

General function of continuous random variable: Let X have density function f_X , and let g be differentiable with derivative equalling 0 at only finitely many points. Let $Y = g(X)$. Then

$$f_Y(y) = \sum_{\substack{x:g(x)=y \\ g'(x) \neq 0}} \frac{f_X(x)}{|g'(x)|}.$$

Ex) $X \sim N(0,1)$. $Y = X^2$.

$$g(x) = x^2 \quad g'(x) = 2x$$

$$\{x : g(x) = y\} = \{\sqrt{y}, -\sqrt{y}\} \quad \text{for } y \geq 0$$

$$\{x : g(x) = y\} = \{\} \quad \text{for } y < 0$$

$$f_Y(y) = \sum_{x \in \{\sqrt{y}, -\sqrt{y}\}} \frac{\phi(x)}{|2x|} \quad \text{for } y \geq 0$$

$$= \frac{\phi(\sqrt{y})}{|2(-\sqrt{y})|} + \frac{\phi(\sqrt{y})}{|2\sqrt{y}|} = \frac{2\phi(\sqrt{y})}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \quad \text{for } y \geq 0$$

$$f_Y(y) = 0 \quad \text{for } y < 0$$

Same as above!