

Section 5.1

Moment generating function

Recall: Distribution of r.v. X is the set of probabilities $\{P(X \in B)\}$, i.e., if you know the distribution, you can always answer “What is $P(X \in B)$?”

Distribution is characterized by:

- Pdf or pmf
- CDF
- Today's topic: Moment generating function (MGF)

Def: the *moment generating function* (mgf) of a r.v. X is the function $M: \mathbb{R} \rightarrow \mathbb{R}^+$ defined by

$$M(t) = M_X(t) = \mathbb{E}e^{tX}, \quad t \in \mathbb{R}.$$

Ex) $X \sim \text{Bern}(p)$.

$$M(t) = \underline{\mathbb{E} e^{tX}} = e^{t \cdot 0} p(0) + e^{t \cdot 1} p(1) = 1(1-p) + e^t p = \boxed{1-p+pe^t}$$

Ex) $X \sim \text{Unif}[0,1]$,

$$M(t) = \underline{\mathbb{E} e^{tX}} = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_0^1 e^{tx} \cdot 1 dx = \frac{e^{tx}}{t} \Big|_0^1$$

$$f(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases} \quad = \frac{e^t - 1}{t} \quad \text{for } t \neq 0, \quad M(0) = 1$$

Ex) $X \sim N(0,1)$ $M(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(\underbrace{x^2 - 2tx}_{(x-t)^2 - t^2})\right) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}[(x-t)^2 - t^2]\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} e^{\frac{t^2}{2}} dx = e^{t^2/2} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} dx}_{P(N(t,1) \in (-\infty, \infty)) = 1}$$

$$= \boxed{e^{t^2/2}}$$

Ex) $X \sim \text{Exp}(\lambda)$

$$M(t) = \mathbb{E}e^{tx} = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{(t-\lambda)x} dx$$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$= \lambda \underbrace{\left. \frac{e^{(t-\lambda)x}}{t-\lambda} \right|_0^{\infty}}_{(*)} = \lambda \frac{0 - 1}{t - \lambda} = \frac{\lambda}{\lambda - t} \quad \text{for } t < \lambda$$

For $t \geq \lambda$, $(*) = \infty \Rightarrow$

$$M(t) = \begin{cases} \infty & t \geq \lambda \\ \frac{\lambda}{\lambda - t} & t < \lambda \end{cases}$$

Using the mgf to generate moments

Recall: $\mathbb{E}X^n$ is the n^{th} moment of X .

Fact: $M^{(n)}(0) = \mathbb{E}X^n, \quad n = 0, 1, 2, 3, \dots$

Pf (for $n = 1$): I.e., we will prove that $M'(0) = \mathbb{E}X$.

$$\begin{aligned} M'(0) &= \left. \frac{d}{dt} M(t) \right|_{t=0} = \left. \frac{d}{dt} \mathbb{E} e^{tX} \right|_{t=0} = \mathbb{E} \left[\left. \frac{d}{dt} e^{tX} \right|_{t=0} \right] \\ &= \mathbb{E} [X e^{tX}] \Big|_{t=0} = \mathbb{E} [X e^{0 \cdot X}] = \mathbb{E} X \quad \checkmark \end{aligned}$$

*requires justification
(dominated conv. thm)*

$$\text{Ex) } X \sim \text{Bern}(p),$$

$$M(t) = pe^t + 1 - p.$$

$$M'(t) = \underline{pe^t}$$

$$\mathbb{E}X = M'(0) = \underline{pe^0 = p} \quad \checkmark$$

$$M''(t) = \underline{\frac{d}{dt} M'(t) = \frac{d}{dt} pe^t = pe^t}$$

$$\mathbb{E}X^2 = M''(0) = \underline{pe^0 = p}$$

$\vdots$

$$M^{(n)}(t) = \underline{pe^t}$$

$$\mathbb{E}X^n = M^{(n)}(0) = \underline{p}$$

Note: $X \in \{0, 1\}$ and $0^n = 0$, $1^n = 1$, so $X^n = X$
 so $\mathbb{E}X^n = \mathbb{E}X = p$ for $n \in \{1, 2, 3, \dots\}$

$$X \sim \text{Exp}(\lambda)$$

$$M(t) = \begin{cases} \infty, & t \geq \lambda \\ \frac{\lambda}{\lambda - t}, & t < \lambda \end{cases}$$

For $t < \lambda$,

$$M'(t) = \frac{\lambda}{(\lambda - t)^2} \quad t < \lambda \quad \text{so } \mathbb{E}X = M'(0) = \frac{\lambda}{(\lambda - 0)^2} = \frac{1}{\lambda}$$

$$M''(t) = \frac{d}{dt} \frac{\lambda}{(\lambda - t)^2} = \frac{2\lambda}{(\lambda - t)^3} \quad \text{so } \mathbb{E}X^2 = M''(0) = \frac{2\lambda}{(\lambda - 0)^3} = \frac{2}{\lambda^2}$$

$$\text{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \boxed{\frac{1}{\lambda^2}} \quad \checkmark$$

Equality in distribution

Def: We say r.v.s X and Y are *equal in distribution* if
$$P(X \in B) = P(Y \in B) \quad \text{for all subsets } B \subset \mathbb{R}.$$

Notation: $X \stackrel{dist}{=} Y$

Fact: If $M_X(t) = M_Y(t) \neq \infty$ for all $t \in \mathbb{R}$, then $X \stackrel{dist}{=} Y$.

What if $M_Y(t) = \infty$ for some t ?

Fact: If there exists $\delta > 0$ so that

$M_X(t) = M_Y(t) \neq \infty$ for all $t \in (-\delta, \delta)$, then $X \stackrel{dist}{=} Y$.

Q) Suppose X has MGF, $M(t) = \begin{cases} \infty, & t \geq 4 \\ \frac{4}{4-t}, & t < 4 \end{cases}$.

What is the distribution of X ?

$X \sim \text{Exp}(4)$ since $\text{Exp}(\lambda)$ r.v. has MGF

$$\begin{cases} \infty & t \geq \lambda \\ \frac{\lambda}{\lambda - t} & t < \lambda \end{cases}$$

Ex) Let $X \sim \text{Exp}(\lambda)$. What is the distribution of $10X$?

$$\text{Mgf of } 10X: M_{10X}(t) = \frac{\mathbb{E} e^{10xt}}{=} = \mathbb{E} e^{x(10t)} \\ = M_X(10t)$$

$$= \begin{cases} \infty & 10t \geq \lambda \\ \frac{\lambda}{\lambda - 10t} & 10t < \lambda \end{cases}$$

$$= \boxed{\begin{cases} \infty & t \geq \lambda/10 \\ \frac{\lambda/10}{\lambda/10 - t} & t < \lambda/10 \end{cases}}$$

So $\boxed{10X \sim \text{Exp}(\lambda/10)}$