

Section 4.5

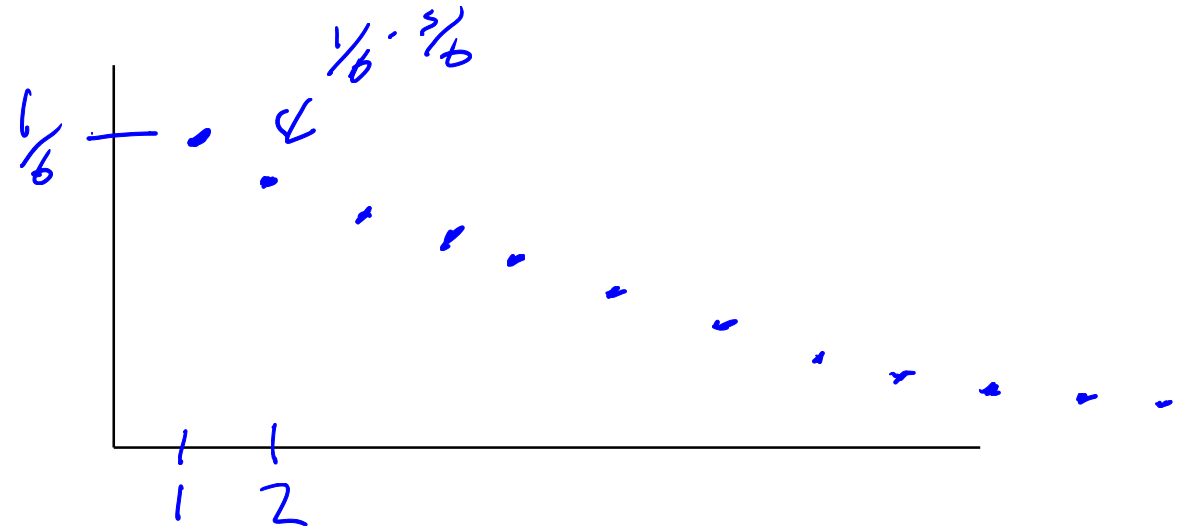
Exponential distribution

Recall: Geometric dist.

Ex) Roll a die until you get a 5. X = number of rolls $\sim$ Geom($\frac{1}{6}$)

Q) What is the most likely value for X ? Ans $X=1$

Pmf: $p(k) = \left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6}$



Exponential dist. models continuous *waiting times*

Q) You are waiting at a bus stop until bus arrives. What is distribution of the time you wait?

A) We need a *continuous* analog of geometric distribution.

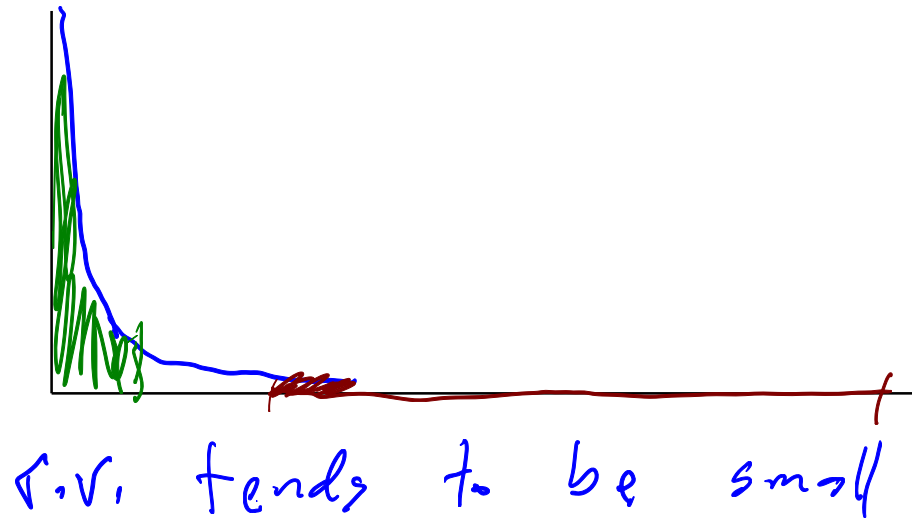
Def: A continuous r.v. X has *exponential dist.* with parameter $\lambda > 0$ if X has density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

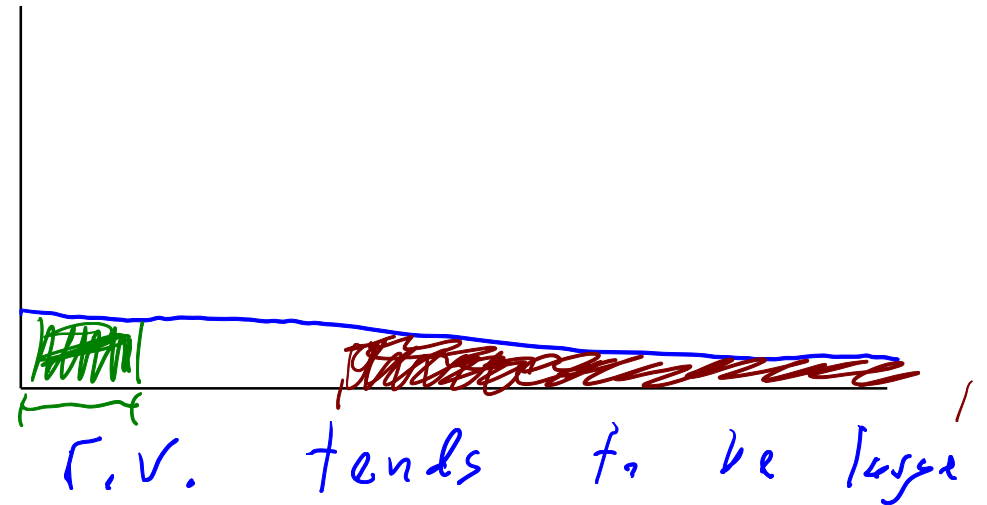
Notation: $X \sim \text{Exp}(\lambda)$, λ is called the *rate*.

Exponential dist. $f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$

λ is large



λ is small



Examples

Q) Let $X \sim \text{Exp}(\lambda)$, $t > 0$. What is $P(X > t)$?

$$P(X > t) = \int_t^{\infty} f(x) dx = \int_t^{\infty} \lambda e^{-\lambda x} dx = -e^{-\lambda x} \Big|_t^{\infty} = 0 - (-e^{-\lambda t}) = e^{-\lambda t}$$

$$P(X > t) = e^{-\lambda t}$$

$$\text{Ex) } X \sim \text{Exp}(1), P(X > 10) = \frac{e^{-1 \cdot 10} = e^{-10}}{}$$

$$Y \sim \text{Exp}\left(\frac{1}{10}\right), P(Y > 10) = \frac{e^{-\frac{1}{10} \cdot 10} = e^{-1}}{}$$

Note: CDF of $X \sim \text{Exp}(\lambda)$ is $F(t) = 1 - P(X > t) = 1 - e^{-\lambda t}$

Thus, $P(X > t)$ characterizes CDF. So if another r.v. Y satisfies $P(Y > t) = e^{-\lambda t}$, then $Y \sim \text{Exp}(\lambda)$.

Q) $X \sim \text{Exp}(\lambda)$. What is the distribution of $60X$?

$$Y = 60X \sim \text{Exp}\left(\frac{\lambda}{60}\right). \quad \text{Proof: } P(Y > t) = P(60X > t) \\ = P(X > \frac{t}{60}) \\ = e^{-\lambda \frac{t}{60}} = e^{-\frac{\lambda}{60}t}$$

Q) $X \sim \text{Exp}(\lambda), Y \sim \text{Exp}(\alpha)$ are independent. What is the distribution of $Z := \min(X, Y)$?

$$P(Z > t) = P(\underbrace{\min(X, Y)}_{\{X > t\} \cap \{Y > t\}} > t) = P(X > t) \cdot P(Y > t) \quad t > 0 \\ = e^{-\lambda t} \cdot e^{-\alpha t} \\ = e^{-(\lambda + \alpha)t}$$

so, $Z \sim \text{Exp}(\lambda + \alpha)$

Properties of $X \sim \text{Exp}(\lambda)$

1. $\mathbb{E}X = \frac{1}{\lambda}$, $\text{Var}(X) = \frac{1}{\lambda^2}$ $\leftarrow$ Exercise

2. $\alpha X \sim \underline{\text{Exp}(\frac{\lambda}{\alpha})}$, for $\alpha > 0$

3. CDF: $F(s) = \begin{cases} \frac{1 - e^{-\lambda s}}{0} & \text{if } s \geq 0 \\ 0 & \text{if } s < 0 \end{cases}$

4. If $Y \sim \text{Exp}(\alpha)$ is indep. of X , then $\min(X, Y) \sim \text{Exp}(\lambda + \alpha)$

5. Memorylessness: "Given that the bus didn't come in the first 10 minutes, the probability that it doesn't come in the next 10 minutes remains unchanged."

$P(X > s + t | X > t) = \underline{P(X > s)}$ for $s, t > 0$

Proof of memorylessness:

$$\begin{aligned}P(X > s + t | X > t) &= \frac{P(X > s + t, X > t)}{P(X > t)} \\&= \frac{P(X > s + t)}{P(X > t)} \\&= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} \\&= e^{-\lambda s - \cancel{\lambda t} + \cancel{\lambda t}} \\&= e^{-\lambda s} = P(X > s) \quad \checkmark\end{aligned}$$

Application example

Ex) Model the lifetime of a lightbulb as exponential with mean 3 years.

1. What is the probability that your new bulb breaks within a year?
2. Your bulb has been working for 2 years. What is the probability that it breaks in the next year?

$$1. X \sim \text{Exp}(\lambda)$$

$$E X = \frac{1}{\lambda} = 3 \Rightarrow \lambda = \frac{1}{3}$$

$$P(X \leq 1) = 1 - P(X > 1) = 1 - e^{-\frac{1}{3} \cdot 1} = \boxed{1 - e^{-\frac{1}{3}}}$$

$$2. P(X \in [2, 3] | X > 2) = P(X \leq 1) \quad \text{by memorylessness}$$
$$= \boxed{1 - e^{-\frac{1}{3}}}$$

Q) Suppose that there are 3 customers waiting in line to be served by one of two cashiers. Both cashiers are currently busy helping a customer. The time it takes to serve a customer is distributed as $Exp(\lambda)$ (and all serving times are independent). As soon as a customer is served, they exit the store.

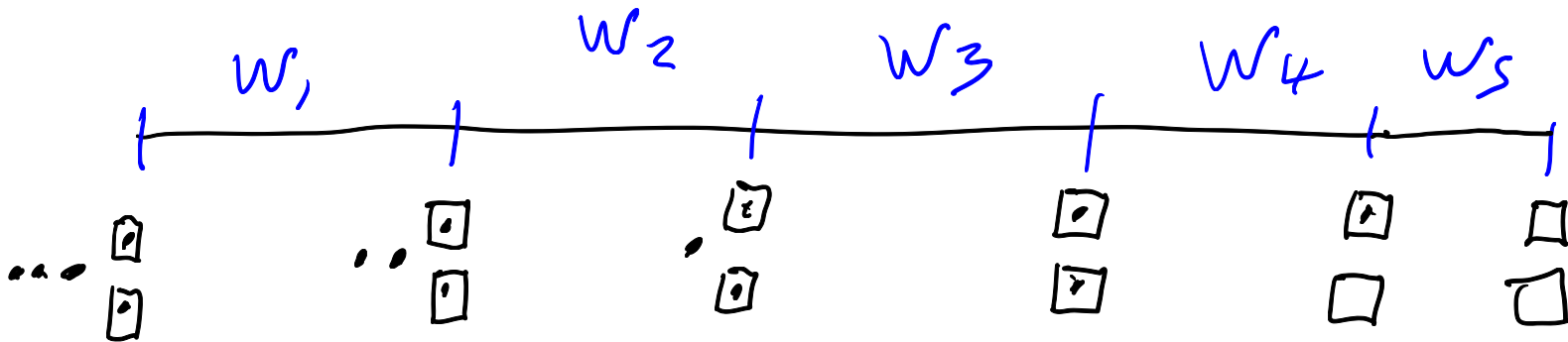
1. What is the distribution of the waiting time for a customer to exit the store.
2. What is the expected value of the time it takes for all five customers to exit the store?

1. $\square$ $W_1 = \text{time for cust. to leave}$
 $\square,$ $= \min(X, Y)$ $X, Y \sim Exp(\lambda)$
1st cashier time to serve 2nd cashier time to serve
 $W_1 \sim Exp(2\lambda)$

2. Let T be time at which last cust. leaves.

$$T = w_1 + w_2 + w_3 + w_4 + w_5$$

$$w_i = (\text{time at which } i\text{th cust. leaves}) \\ - (\text{time at which } (i-1)\text{st cust. leaves})$$



By memory lessness $w_2 \stackrel{\text{dist}}{=} w_1$

Similarly $w_3 \stackrel{\text{dist}}{=} w_4 \stackrel{\text{dist}}{=} w_1$,

$$w_5 \sim \text{Exp}(\lambda)$$

$$\mathbb{E} T = \mathbb{E} w_1 + \mathbb{E} w_2 + \mathbb{E} w_3 + \mathbb{E} w_4 + \mathbb{E} w_5$$

$$= 4 \cdot \mathbb{E} [\text{Exp}(2\lambda)] + \mathbb{E} [\text{Exp}(\lambda)]$$

$$= \frac{4}{2\lambda} + \frac{1}{\lambda} = \boxed{\frac{3}{\lambda}}$$