

Section 10.2

Conditional distribution for jointly continuous r.v.s
(Cliff notes: Analogous to discrete case)

Recall: Discrete case

For discrete r.v.s X, Y :

- Conditional pmf: $\overbrace{p_{X|Y}(x|y)}^{P(X=x|Y=y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$

- Conditional expectation: $\mathbb{E}[X|Y = y] = \sum_x x p_{X|Y}(x|y)$

- $P_X(x) = \sum_y \overbrace{P_{X|Y}(x|y)}^{P_{X,Y}(x,y)} P_Y(y)$

$$\mathbb{E}[g(X)|Y = y] = \frac{\sum_x g(x) P_{X|Y}(x|y)}{1}$$

- $\mathbb{E}[X] = \sum_y \mathbb{E}[X|Y=y] P_Y(y)$

Continuous case: Analogous to discrete

Def: Let X, Y be jointly continuous. The *conditional density function* of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

The *conditional expectation* of $g(X)$ given $Y = y$ is

$$\mathbb{E}[g(x)|Y = y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y) dx$$

The *conditional probability* that $X \in A$, given $Y = y$ is

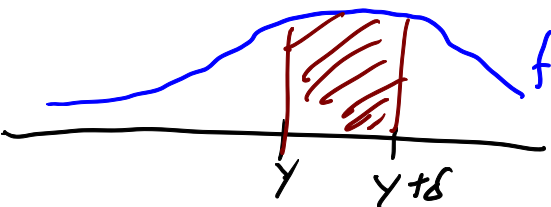
$$P(X \in A|Y = y) = \int_A f_{X|Y}(x|y) dx$$

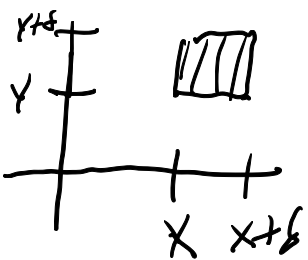
- Defined for y such that $f_Y(y) > 0$.
- Looks like Bayes rule, but isn't a direct application since $P(Y = y) = \underline{0}$

Intuition for definition

Claim: $f_{X|Y}(x|y) \approx \frac{P(X \in [x, x+\delta] | Y \in [y, y+\delta])}{\delta}$

RHS: $\frac{P(X \in [x, x+\delta], Y \in [y, y+\delta])}{P(Y \in [y, y+\delta]) \cdot \delta} = \frac{a}{b \delta}$

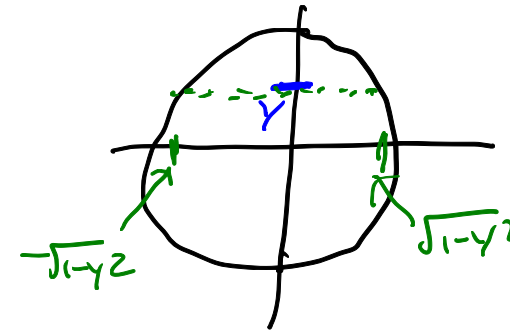
$b = P(Y \in [y, y+\delta]) =$  $\int_y^{y+\delta} f_Y(y) \approx f_Y(y) \cdot \delta$

$a = P((X, Y) \in [x, x+\delta] \times [y, y+\delta]) =$  $= f_{X,Y}(x,y) \cdot \delta^2$

$\Rightarrow \frac{a}{b \delta} = \frac{f_{X,Y}(x,y) \cdot \delta^2}{f_Y(y) \cdot \delta \cdot \delta} = f_{X|Y}(x|y) \quad \checkmark$

Ex) Let $X, Y \sim \text{Unif}(B)$, where $B = \{(x, y): x^2 + y^2 \leq 1\}$. Let $y \in (-1, 1)$. What is the conditional distribution and expectation of X given $Y = y$?

$$f_{X|Y}(x|y) = \frac{\overbrace{f_{X,Y}(x,y)}^{f(x,y)}}{f_Y(y)}$$



$$f(x,y) = \begin{cases} 1/\pi & (x,y) \in B \\ 0 & (x,y) \notin B \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y) dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx = \frac{2\sqrt{1-y^2}}{\pi}$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{1/\pi}{2\sqrt{1-y^2}} & (x,y) \in B \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{2\sqrt{1-y^2}} & x \in [-\sqrt{1-y^2}, \sqrt{1-y^2}] \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow X|Y=y \sim \text{Unif}[-\sqrt{1-y^2}, \sqrt{1-y^2}]$$

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx = \mathbb{E}[\text{Unif}[-\sqrt{1-y^2}, \sqrt{1-y^2}]] = 0$$

Averaging conditional expectations

Note: By definition $f_{X,Y}(x,y) = f_{X|Y}(x|y) \cdot \underline{f_Y(y)}$

Thus,

Fact: Let X, Y be jointly continuous, then

$$f_X(x) = \underline{\int_{-\infty}^{\infty} f_{X,Y}(x,y) dy} = \int_{-\infty}^{\infty} f_{X|Y}(x|y) \cdot f_Y(y) dy$$

and

$$\mathbb{E}g(X) = \underline{\int_{-\infty}^{\infty} \mathbb{E}[g(X)|Y=y] \cdot f_Y(y) dy}$$

- Assuming conditional expectations are well defined.
- If $f_Y(y) = 0$, integrand is 0 (not undefined).

Bayesian statistics

Unknown arg weight is random

Ex) Suppose that you find a new species of bat and model the weight of a particular bat as $X \sim N(Y, 1)$ kilograms where **Y is random** and $Y \sim N(0, 1)$. (Assuming a universe that allows negative weight...) You measure the weight of one bat to be 2 kilograms. What is the probability that $Y \leq 1$?

$$f_Y(Y) = \phi(Y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Y^2}$$

$$X|Y=y \sim N(y, 1), \quad f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-y)^2} = \phi(x-y)$$

Condition on $X=2$

$$P(Y \leq 1 | X=2) = ?$$

$$P(Y \leq 1 | X=x) = \int_{-\infty}^1 \underbrace{f_{Y|X}(y|x)}_{\text{need to find } f_{Y|X}(y|x)} dy$$

$$f_{Y|X}(y|x) = \frac{f_{Y,X}(y,x)}{f_X(x)} = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\overbrace{f_{X|Y}(x|y)}^{\phi(x-y)} \cdot \underbrace{f_Y(y)}_{\phi(y)}}{f_X(x)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy = \int_{-\infty}^{\infty} f_{X|Y}(x|y) \cdot f_Y(y) dy = \int_{-\infty}^{\infty} \phi(x-y) \cdot \phi(y) dy = \phi * \phi(x)$$

$$\text{Recall: } N(0,1) + N(0,1) = N(0,2) \Rightarrow X \sim N(0,2)$$

$$\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi} \cdot 2} \cdot e^{-\frac{1}{2} \frac{x^2}{2}}$$

Plug in above:

$$f_{Y|X}(y|x) = \frac{\phi(x-y) \cdot \phi(y)}{f_X(x)} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-y)^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}}{\frac{1}{\sqrt{2\pi \cdot 2}} \cdot e^{-\frac{1}{2} \frac{x^2}{2}}}$$

$$= \frac{1}{\sqrt{2\pi} \cdot \frac{1}{2}} \exp \left[-\frac{1}{2} \underbrace{(x-y)^2 + y^2 - \frac{x^2}{2}}_a \right]$$

$$a = x^2 - 2xy + y^2 + y^2 - \frac{x^2}{2} = 2 \left(y^2 - xy + \frac{x^2}{4} \right) = 2 \left(y - \frac{x}{2} \right)^2$$

$$\Rightarrow f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi} \cdot \frac{1}{2}} \cdot \exp \left(-\frac{1}{2} \left[\frac{(y - \frac{x}{2})^2}{\frac{1}{2}} \right] \right) \quad \text{i.d.}$$

$$Y|X=x \sim N\left(\frac{x}{2}, \frac{1}{2}\right)$$

$$\Rightarrow P(Y \leq 1 | X=2) = P\left(N\left(\frac{2}{2}, \frac{1}{2}\right) \leq 1\right) = P(N(0, \frac{1}{2}) \leq 0) = \boxed{\frac{1}{2}}$$