

10.1

Conditional distribution of discrete r.v.

Ex) Flip 3 coins and let X be the number of heads. $X \sim \text{Bin}(3, \frac{1}{2})$

Now, I tell you that the first coin is heads. What is the new (conditional) distribution of X ?

$1 + \text{Bin}(2, \frac{1}{2})$
↑
1st coin is H remaining 2 coins are still rand.

Now, instead, I tell you that heads never appears twice in a row. What is the conditional distribution of X ?

We need to find the conditional probabilities: $\underline{P(X = k | \text{No HH})}$

Flipped 3 coins: $\Omega = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$

$B = \{\text{no HH}\} = \{\underline{TTT, TTH, THT, HTT, HTH}\}$

Conditional pmf:

$$P(X = 0|B) = \frac{P(X=0,B)}{P(B)} = \frac{P(\{TTT\})}{P(B)} = \frac{1/8}{5/8} = 1/5$$

$$P(X = 1|B) = \frac{P(X=1,B)}{P(B)} = \frac{P(\{TTH, THT, HTT\})}{P(B)} = \frac{3/8}{5/8} = \frac{3}{5}$$

$$P(X = 2|B) = \frac{P(X=2,B)}{P(B)} = \frac{P(\{HTH\})}{P(B)} = \frac{1/8}{5/8} = 1/5$$

$$P(X = 3|B) = \frac{P(X=3,B)}{P(B)} = \frac{P(\{\})}{P(B)} = 0$$

Conditional distribution of discrete r.v.

Def: Let X be a discrete r.v. and B be an event with $P(B) > 0$. The conditional pmf of X given B is defined as

$$P_{X|B}(k) = P(X = k|B) = \frac{P(X = k, B)}{P(B)} \quad \text{for v.l.s } k \text{ that } X \text{ can take}$$

In example above, $P_{X|B}(0) = \frac{1}{5}$, $P_{X|B}(1) = \frac{3}{5}$, $P_{X|B}(2) = \frac{1}{5}$
 $P_{X|B}(3) = 0$

Ex) Flip 3 coins, let $X = \text{sum}$. What is the conditional pmf of X given $X \geq 1$?

of heads.

$X \sim \text{Bin}(3, 1/2)$

pmf of X : $P_X(0) = 1/8$, $P_X(1) = 3/8$, $P_X(2) = 3/8$, $P_X(3) = 1/8$

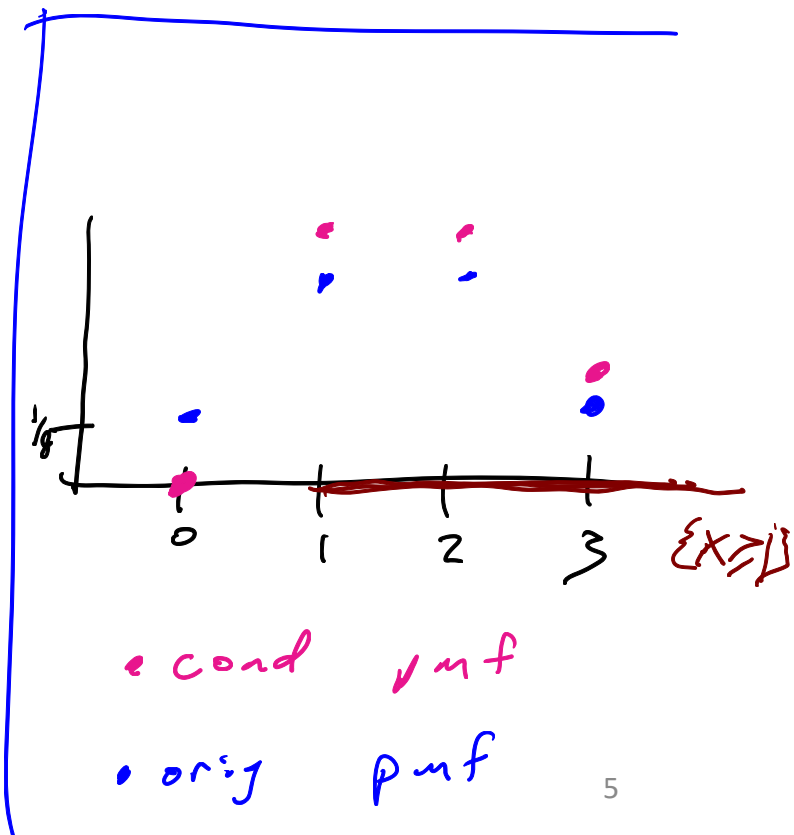
$$P(X \geq 1) = 1 - P_X(0) = 7/8$$

$$P_{X|X \geq 1}(0) = \frac{P\{X=0, X \geq 1\}}{P\{X \geq 1\}} = \frac{0}{7/8} = 0$$

$$P_{X|X \geq 1}(1) = \frac{P\{X=1, X \geq 1\}}{P\{X \geq 1\}} = \frac{P\{X=1\}}{P\{X \geq 1\}} = \frac{3/8}{7/8} = \frac{3}{7}$$

$$P_{X|X \geq 1}(2) = \frac{P\{X=2, X \geq 1\}}{P\{X \geq 1\}} = \frac{P\{X=2\}}{P\{X \geq 1\}} = \frac{3/8}{7/8} = \frac{3}{7}$$

$$P_{X|X \geq 1}(3) = \frac{P\{X=3, X \geq 1\}}{P\{X \geq 1\}} = \frac{P\{X=3\}}{P\{X \geq 1\}} = \frac{1/8}{7/8} = \frac{1}{7}$$



Conditional expectation

Def: Let X be a discrete r.v. and B be an event with $P(B) > 0$. The conditional expectation of X given B is defined as

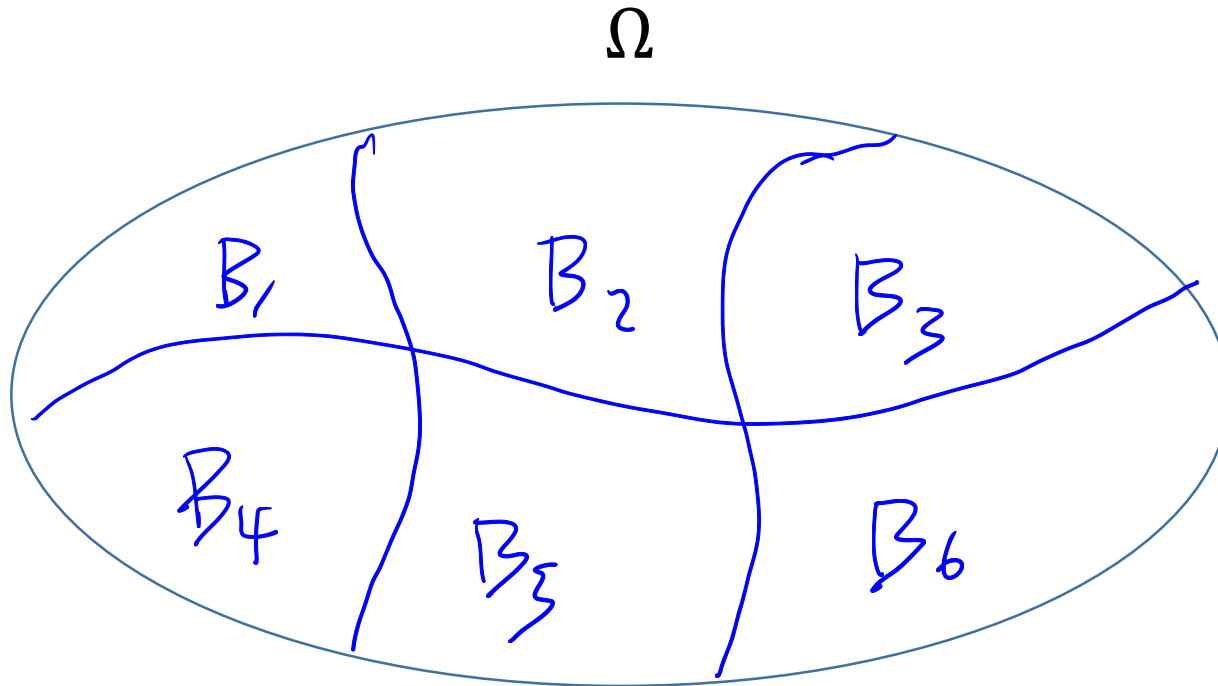
$$\mathbb{E}[X|B] = \frac{\sum_k k P_{X|B}(k)}{P(B)}$$

Ex) $X \sim \text{Bin}\left(3, \frac{1}{2}\right)$. $\mathbb{E}[X|X \geq 1] = 0 \cdot P_{X|X \geq 1}(0) + 1 \cdot P_{X|X \geq 1}(1) + 2 \cdot P_{X|X \geq 1}(2) + 3 \cdot P_{X|X \geq 1}(3)$

$$= 0 \cdot 0 + 1 \cdot \frac{3}{7} + 2 \cdot \frac{3}{7} + 3 \cdot \frac{1}{7}.$$

Recovering distribution from conditional distribution(s)

Recall: Partition



$$\bigcup_i B_i = \Omega$$

$$B_i \cap B_j = \{\emptyset\}, \quad i \neq j$$

Fact: Let $B_1, B_2, \dots, B_n$ be a partition of Ω with $P(B_i) > 0$. Then

$$p_X(k) = \underline{P(X=k) = \sum_{i=1}^n P(X=k | B_i) \cdot P(B_i)} \leftarrow \text{law of total prob.}$$
$$= \sum_{i=1}^n P_{X|B_i}(k) \cdot P(B_i)$$

$$\mathbb{E}X = \underline{\sum_{i=1}^n \mathbb{E}[X | B_i] P(B_i)}$$

Ex) Let X be the number of customers at an ice cream shop.

On rainy days, the number of customers $\sim \text{Poisson}(\alpha)$, $\leftarrow X|R \sim \text{Poisson}(\alpha)$

On sunny days, the number of customers $\sim \text{Poisson}(\beta)$, $\leftarrow X|S \sim \text{Poisson}(\beta)$

$P(\underbrace{\text{rainy}}_R) = 0.4$, $P(\underbrace{\text{sunny}}_S) = 0.6$.

Pmf of X ?

$$P_X(k) = P_{X|R}(k) \cdot P(R) + P_{X|S}(k) \cdot P(S) \\ = P(\text{Poisson}(\alpha) = k) \cdot 0.4 + P(\text{Poisson}(\beta) = k) \cdot 0.6$$

$$= \left[\frac{e^{-\alpha} \cdot \alpha^k}{k!} \cdot 0.4 + \frac{e^{-\beta} \cdot \beta^k}{k!} \cdot 0.6 \right]$$

$$k \in \{1, 2, \dots\}$$

$$\begin{aligned}
 \mathbb{E}X &= \underline{\mathbb{E}[X|R] \cdot P(R) + \mathbb{E}[X|S] \cdot P(S)} \\
 &= \mathbb{E}[\text{Poisson}(\lambda)] \cdot 0.4 + \mathbb{E}[\text{Poisson}(\beta)] \cdot 0.6 \\
 &= \boxed{2 \cdot 0.4 + \beta \cdot 0.6}
 \end{aligned}$$

Conditioning on event $\rightarrow$ conditioning on r.v.

Conditioning on events, we just found

- conditional pmf,
- conditional expectation,
- How to go from conditional pmf/expectations on a set of events which makes a partition, to unconditional pmf/expectation

Now we will do the same thing with the events $\{Y = y\}$, thus conditioning on r.v. Y .

Conditioning on a r.v.

Let Y be a r.v. We wish to condition on event $\{Y = y\}$.

Def: For discrete r.v.s X, Y , the *conditional pmf* of X given $Y = y$ is the 2-variable function

$$p_{X|Y}(x|y) = \frac{P(X=x | Y=y)}{P(Y=y)} = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

The conditional expectation of X given $Y = y$ is

$$\mathbb{E}[X|Y = y] = \sum_x x \cdot p_{X|Y}(x|y)$$

Ex) Let X, Y have joint pmf given in the following table:

		Y	
		0	1
X	0	1/10	2/10
	1	3/10	4/10

$|P_Y(0) = \frac{3}{10}|$
 $|P_Y(1) = \frac{6}{10}|$

Find $p_{X|Y}(x|y)$ for all $x, y \in \{0, 1\}$.

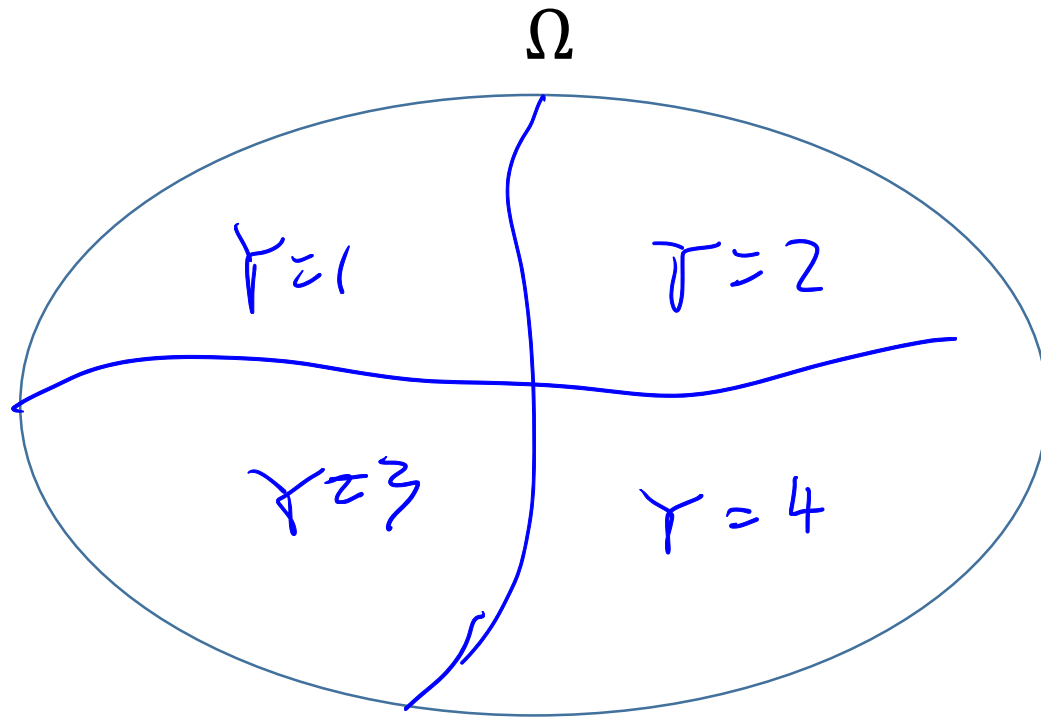
$$P_{X|Y}(x|y) = \frac{P(x, y)}{P_Y(y)}$$

$$\begin{aligned}
 P_{X|Y}(0|0) &= \frac{P(0, 0)}{P_Y(0)} = \frac{1/10}{4/10} = \frac{1}{4} \\
 P_{X|Y}(1|0) &= \frac{P(1, 0)}{P_Y(0)} = \frac{3/10}{4/10} = \frac{3}{4}
 \end{aligned}
 \left. \vphantom{\begin{aligned} P_{X|Y}(0|0) \\ P_{X|Y}(1|0) \end{aligned}} \right\} \text{sum to 1}$$

$$\begin{aligned}
 P_{X|Y}(0|1) &= \frac{P(0, 1)}{P_Y(1)} = \frac{2/10}{6/10} = \frac{1}{3} \\
 P_{X|Y}(1|1) &= \frac{P(1, 1)}{P_Y(1)} = \frac{4/10}{6/10} = \frac{2}{3}
 \end{aligned}
 \left. \vphantom{\begin{aligned} P_{X|Y}(0|1) \\ P_{X|Y}(1|1) \end{aligned}} \right\} \text{sum to 1}$$

Partition defined by $\{Y = y\}$ events

The events $\{Y = y\}$, for all values Y can take, form a partition.



Thus,

Fact:

$$p_X(x) = \sum_y P_{X|Y}(x|y) \cdot P_Y(y)$$

$$\mathbb{E}X = \sum_y \mathbb{E}[X|Y=y] \cdot P_Y(y)$$

Ex) $Z = X + Y$, $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\alpha)$ be independent.
Find the conditional pmf of X given $Z = n$.

Recall: $Z \sim \text{Poisson}(\lambda + \alpha)$

$$P(Z=n) = \frac{e^{-(\lambda+\alpha)} \cdot (\lambda+\alpha)^n}{n!}$$

$$P_{X|Z}(k|n) = P(X=k | Z=n) = \frac{P(X=k, Z=n)}{P_Z(n)}$$

$$= \frac{P(X=k, Y=n-k)}{P_Z(n)}$$

$$k \in \{0, 1, \dots, n\}$$

$$= \frac{P(X=k) \cdot P(Y=n-k)}{P_Z(n)}$$

by indep.

$$= \frac{\frac{e^{-\lambda} \cdot \lambda^k}{k!} \cdot \frac{e^{-\alpha} \cdot \alpha^{n-k}}{(n-k)!}}{\frac{e^{-(\lambda+\alpha)} \cdot (\lambda+\alpha)^n}{n!}}$$

$$= \frac{n!}{k!(n-k)!} \cdot \frac{\lambda^k \cdot \alpha^{n-k}}{(\lambda+\alpha)^n} = \binom{n}{k} \cdot \underbrace{\left(\frac{\lambda}{\lambda+\alpha}\right)^k}_{p^k} \cdot \underbrace{\left(\frac{\alpha}{\lambda+\alpha}\right)^{n-k}}_{(1-p)^{n-k}}$$

$$\text{set } p = \frac{\lambda}{\lambda+\alpha} \quad \text{so} \quad 1-p = 1 - \frac{\lambda}{\lambda+\alpha} = \frac{\alpha}{\lambda+\alpha}$$

$$\Rightarrow \left\{ X | \{Z=n\} \sim \text{Bin} \left(n, \frac{\lambda}{\lambda+\alpha} \right) \right\}$$

Sanity check:

$$\mathbb{E} X = \sum_n \mathbb{E}[X | Z=n] \cdot P(Z=n) = \sum_{n=0}^{\infty} \underbrace{\mathbb{E} \left[\text{Bin} \left(n, \frac{\lambda}{\lambda+\alpha} \right) \right]}_{n \cdot \frac{\lambda}{\lambda+\alpha}} \cdot P_Z(n)$$

$$= \frac{\lambda}{\lambda+\alpha} \underbrace{\sum_{n=0}^{\infty} n \cdot P_Z(n)}_{\mathbb{E} Z} = \frac{\lambda}{\lambda+\alpha} \cdot \underbrace{\mathbb{E} [\text{Poisson}(\lambda+\alpha)]}_{\lambda+\alpha} = \frac{\lambda}{\lambda+\alpha} \cdot (\lambda+\alpha) = \boxed{\lambda} \quad \checkmark$$