Review: Remainder Theorem, Rolle's Theorem, Optimization word problems, Mean Value Theorem

All problems come from previous final exams (years 2005-2012)

1. Give a complete proof that for all x satisfying $-1 \leq x \leq 1$,

$$0 \leq cos(x) - \left(1 - \frac{x^2}{2}\right) \leq \frac{1}{24}.$$

2. Let $T_2(x)$ be the second degree Taylor polynomial about $a = 8$ for $f(x) = x^{1/3}$.

   (a) Find $T_2(x)$ and simplify your answer.

   (b) is $T_2(8.1)$ larger than $8.1^{1/3}$? Justify your answer.

3. if $24m^2$ of material is available to make a rectangular storage container with an open top, and if the length of its base is twice the width, find the largest possible volume of the rectangular storage container. Justify that your answer gives indeed the largest possible volume.

4. If $f(0) = 10$ and $f'(x) \geq 3$ for $0 \leq x \leq 4$, what is the least $f(4)$ could possibly be?

5. Find (with justification), the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 15 - x^2$.

6. Determine what degree Maclaurin polynomial for $ln(1-x)$ that should be used to approximate $ln(1.1)$, so that the approximation is guaranteed to be accurate to within 10^{-9}.

7. (a) Prove that $x + ln|x| = 0$ has at least one solution in the open interval $(-1, 1)$.
   (b) Prove that $x + ln|x| = 0$ has exactly one solution in the open interval $(-1, 1)$.

8. A cylindrical can without a top is made to contain $27\pi cm^3$ of liquid. Determine (with justification) the dimensions of the can that minimize the area of the metal used to make the can.

9. A function $f(x)$ has third derivative equal to $10/(1-x)$. The second-degree Maclaurin polynomial $T_2(x)$ is used to approximate $f(0.1)$. Find the upper bound for the error $|f(0.1) - T_2(0.1)|$.