The University of British Columbia
Math 303: Section 201
2018, March 21

Name: ____________________________  Student ID: __________________

Instructions

• This exam consists of 3 questions worth a total of 40 points.

• Make sure this exam has 5 pages excluding this cover page.

• Explain your reasoning thoroughly, and justify your answers unless the question indicates otherwise.

• No notes, calculators, or other electronic devices are allowed.

• If you need more space, use the back of the pages.

• Duration: 50 minutes.

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1. Consider a three-server system in which a customer is served first by Server 1, then by Server 2, then by Server 3, and then departs. The service times at Server i are exponential random variables with rates $\mu_i, i = 1, 2, 3$.

(a) Suppose that when you arrive, there is no one in the system. What is the expected time until you leave Server 3?

**Solution:**

$$\mathbb{E}[\text{Exp}(\mu_1) + \text{Exp}(\mu_2) + \text{Exp}(\mu_3)] = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3}.$$

(b) Suppose that when you arrive, you find five customers at Server 2, four in line and one being served. What is the probability that at least one customer is still in Server 2 when you get to Server 2?

**Solution:**

Consider the complement of this event — that all of the servers leave Server 2 before you arrive. First, the first customer must leave Server 2 before you finish Server 1. This has probability

$$P(\text{Exp}(\mu_2) < \text{Exp}(\mu_1)) = \frac{\mu_2}{\mu_1 + \mu_2}.$$

Then the next customer must leave before you and so on. By memorylessness, the probability of all 5 leaving before you arrive is

$$\left(\frac{\mu_2}{\mu_1 + \mu_2}\right)^5,$$

and thus the solution is

$$1 - \left(\frac{\mu_2}{\mu_1 + \mu_2}\right)^5.$$
2. Let \( \{N(t) : t \geq 0\} \) be a rate-\( \lambda \) Poisson process. Let \( T_1, T_2, \ldots \) be the interarrival times between events (\( T_1 \) is the arrival of the first event). Let \( s, t > 0 \).

(a) What is \( E[N(t + s) | N(t) = k] \)?

**Solution:**
Recall that \( N(t + s) - N(t) \) is independent of \( N(t) \). Thus,

\[
E[N(t + s) | N(t) = k] = E[N(t + s) - N(t) | N(t) = k] + E[N(t) | N(t) = k]
\]

\[= E[Poisson(\lambda s)] + k = \lambda s + k.\]

(b) What is \( E[N(t) | N(t + s) = k] \)?

**Solution:**
Recall that conditional on \( N(t + s) = k \), \( N(t) \sim Binomial(k, t/(t + s)) \). Thus, the solution is \( kt/(t + s) \).
(c) What is $E[T_1 \mid N(1) = 2]$?

**Solution:**  
Here and below, let $U_1, U_2 \sim Uniform(0, 1)$.

$$E[T_1 \mid N(1) = 2] = E_{\min}(U_1, U_2).$$

Note that for $0 < t < 1$, $P(\min(U_1, U_2) > t) = (1 - t)^2$. Thus,

$$E_{\min}(U_1, U_2) = \int_0^1 P(\min(U_1, U_2) > t) dt = \int_0^1 (1 - t)^2 dt = 1/3.$$

(d) What is $E[T_3 \mid N(1) = 2]$?

**Solution:**  
We split $T_3$ up into the time before 1 and after 1: $T_3 = (1 - S_2) + (S_3 - 1)$. We have

$$E[1 - S_2 \mid N(1) = 2] = E[2 - \max(U_1, U_2)] = E[\min(U_1, U_2)] = 1/3$$

where the second equality follows by symmetry. Also, by memorylessness,

$$E[S_3 - 1 \mid N(1) = 2] = E Exp(\lambda) = 1/\lambda.$$

Thus, the solution is

$$1/3 + 1/\lambda.$$
(e) What is $E[T_1 \mid N(2) - N(1) = 1]$?

**Solution:**

This is tricky. We have

$$
E[T_1 \mid N(2) - N(1) = 1] = E[T_1 \mid N(2) - N(1) = 1, T_1 < 1]P(T_1 < 1\mid N(2) - N(1) = 1) + E[T_1 \mid N(2) - N(1) = 1, T_1 \geq 1]P(T_1 \geq 1\mid N(2) - N(1) = 1).
$$

We start with the first term. By independence of increments,

$$
E[T_1 \mid N(2) - N(1) = 1, T_1 < 1]P(T_1 < 1\mid N(2) - N(1) = 1) = E[T_1 \mid T_1 < 1]P(T_1 < 1) = \int_0^1 \lambda te^{-\lambda t} \, dt = \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{\lambda}.
$$

Now the second term. The key is to see that conditional on the event that $T_1 > 1$ and $N(2) - N(1) = 1$, the time at which $T_1$ occurs follows the distribution $1 + U_1$. Again by independence of increments,

$$
E[T_1 \mid N(2) - N(1) = 1, T_1 \geq 1]P(T_1 \geq 1\mid N(2) - N(1) = 1) = E[T_1 \mid T_1 \geq 1]P(T_1 > 1) = (1 + \mathbb{E}[U_1])e^{-\lambda} = \frac{3}{2}e^{-\lambda}.
$$

Add the two terms to give the solution:

$$
\frac{1 - e^{-\lambda} + \lambda e^{-\lambda}/2}{\lambda}.
$$
3. Starting at 6am, hikers arrive at a mountain according to a Poisson process with rate 3 per hour, and they immediately start hiking the trail. Hikers stop coming to the mountain at 2pm (8 hours later). The sun sets at 6pm (4 more hours later). The length of time, in hours, that a hiker takes to complete the trail is an independent copy of a Uniform[2, 10] random variable. Let $N(t)$ be the number of hikers on the trail after $t$ hours from 6am.

(a) What is the distribution of $N(8)$, the number of hikers still on the trail at 2pm? Give the name and parameter(s).

Solution: Let $U$ be a Uniform[2, 10] random variable.

$$P_s = P(U > 8 - s) = \begin{cases} \frac{2+s}{8} & \text{if } 0 \leq s \leq 6, \\ 1 & \text{if } 6 < s \leq 8. \end{cases}$$

We know that $N(8)$ is Poisson with parameter

$$3 \int_0^8 P_s \, ds = 3 \left( \int_0^6 \frac{2+s}{8} \, ds + 2 \right) = 3(15/4 + 2) = \frac{69}{4}.$$ 

(b) What is the distribution of $N(12)$, the number of hikers still on the trail at 6pm? Give the name and parameter(s).

Solution: Similarly to a) the probability that someone who arrives after $s$ hours of 6am will be on the trail at 6pm is

$$P_s = \begin{cases} 0 & \text{if } 0 \leq s \leq 2, \\ \frac{s-2}{8} & \text{if } 2 < s \leq 8, \\ 0 & \text{if } s > 8. \end{cases}$$

For $s > 8$ we took into account that no hiker arrives after 2pm. Then $N(12)$ is Poisson with parameter

$$3 \int_0^{12} P_s \, ds = 3 \int_2^8 \frac{s-2}{8} \, ds = \frac{27}{4}.$$