Singular Value Decomposition. Part 2 Monday, Nov. 25th/19

Recall: Sum of rank-1 summands version of SVD

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T, \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0.$$

Q: Put matrices in above form:

i) $A = I$

$$I = \sum_{i=1}^{r} (1) e_i e_i^T = \sum_{i=1}^{r} u_i u_i^T, \quad u_1, \dots u_n \text{ are any orthobasis}$$

ii)
$$\begin{bmatrix} 2 & & \\ & 0 & \\ & & 3 \end{bmatrix} = 2e_1 e_1^T + 3e_3 e_3^T$$
$$= 3e_3 e_3^T + 2e_1 e_1^T \rightarrow \text{since } \sigma_1 \geq \sigma_2$$

iii)
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \sqrt{6} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

iv)
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} = 2\left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} e_1^T + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} e_2^T \right) \qquad \sigma_1 = \sigma_2 = 2$$

orthonormal

Note: In general, writing in SVD is complicated.

Two equivalent forms of SVD:

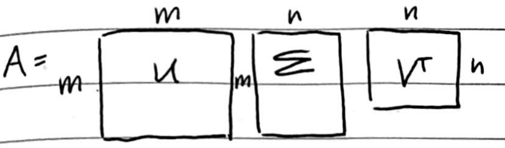
Reduced SVD:

$$A = \boxed{U} \boxed{\Sigma} \boxed{V^T} \in \mathbb{R}^{m \times n}$$

$$U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_r \\ | & | & & | \end{bmatrix} \quad V = \begin{bmatrix} v_1, & \cdots & v_r \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}$$

Full SVD:

Case I. $m \geq n$



$U = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix}$, $V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ $\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & 0 & \end{bmatrix}$

Note: If rank(A) < n, full SVD contains singular values equal to 0.

Implication: ~~U & V will have~~ Σ will have extra "zero" rows.

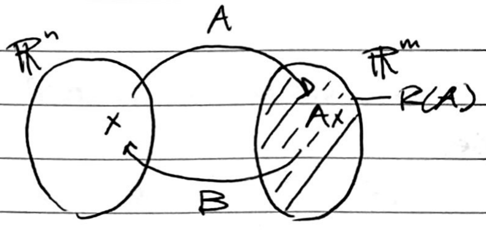
$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$, $\sigma_{r+1} \dots \sigma_n = 0$.

Case II: $m < n$.

$$\Sigma = \begin{array}{c} \\ m \end{array} \overset{n}{\begin{bmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_m & \end{bmatrix}} \quad \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0, \sigma_{r+1}, \dots \sigma_m = 0$$

* w/ complex numbers, $V^T \to$ conjugate transpose.

Left Inverse



Def A left inverse of inverse V, $B \in \mathbb{R}^{n \times m}$, of matrix $A \in \mathbb{R}^{m \times n}$ satisfies $BA = I \in \mathbb{R}^{n \times n}$

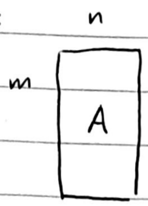
What property does A need to have B?

Prop: A has a left inverse if the following equivalent conditions are satisfied:

$$\left.\begin{array}{l} \text{① } N(A) = \{0\} \\ \text{② Cols of A = lin. indep.} \\ \text{③ rank}(A) = n,\ m > n \end{array}\right\} \text{A is 1-to-1.}$$

Note: This requires $m \geq n$, otherwise cols can't be lin. indep.

Tall matrices:



Great thing about SVD: reduces case to diagonal matrix (because of the simplicity of the orthogonal matrices U & V).

Finding left inverse of A? Look at diagonal.

Goal: Characterize left-inverse.

Warm-up: $A = \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & 0 & \end{bmatrix}$, $\sigma_1, \dots \sigma_n > 0$.

Left inverse: Similarly to A^{-1}, take reciprocal of σ + take transpose:

$$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & & \\ & 1/\sigma_2 & & 0 \\ & & \ddots & \\ & & 1/\sigma_n & \end{bmatrix}, \quad \Sigma^+\Sigma = I$$

↳ although zeros, can be anything in $\mathbb{R}$, making Σ^+ not unique.

In general,

Def: the Moore-Penrose pseudo-inverse of A is:

$$A^+ = V\Sigma^+U^T, \quad \Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & & \\ & \ddots & & 0 \\ & & 1/\sigma_n & \end{bmatrix}$$

Rmk: We assume that $\operatorname{rank}(A) = n$, and thus $\sigma_1, \dots \sigma_n > 0$.

Application to music: Left inverse is used (σ/A) to kill noise.

Some properties:

① $A^+A = I$

↳ $A^+A = V\Sigma^+ \underbrace{U^T U}_{I} \Sigma V^T$

$= V \underbrace{\Sigma^+ \Sigma}_{I} V^T$

$= \cancel{VV^T}$

$= I.$

② To get the set of all left inverses, just replace Σ^+ w/ $\begin{bmatrix} 1/\sigma_1 & & & \\ & \ddots & & C \\ & & 1/\sigma_n & \end{bmatrix}$ any scalars.

③ $AA^+ = A(A^TA)^{-1}A^T = P =$ projection onto $R(A)$.
$AA^+ \neq I$ since $r \times m$ matrix, * $\Sigma\Sigma^+$

④ $A^+ = (A^TA)^{-1}A^T$

Pf. of full SVD given A has full rank.

$A \in \mathbb{R}^{m \times n}$, $A \overset{?}{=} U \Sigma V^T$

Overall: Spectral Thm & Backengineering.

Spectral Thm on A^TA:
Note that A^TA is positive semi-definite (PSD), indeed $x^TA^TAx = \|Ax^2\| \geq 0$.
Thus, all eigvals ≥ 0.
$\Rightarrow A^TA = VDV^T$

$D: \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, $\lambda_1, \dots \lambda_n \geq 0$
since A has full rank ($\lambda_n \neq 0$).

$D^{-1/2} = \begin{bmatrix} 1/\sqrt{\lambda_1} & & & \\ & 1/\sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & 1/\sqrt{\lambda_n} \end{bmatrix}$

∴ purpose: cancel out Σ & V to give U = an orthogonal matrix, $D^{-1/2} \to$ to cancel out Σ.

Then, we will show that $AVD^{-1/2}$ has orthonormal cols.

Indeed, $(AVD^{-1/2})^TAVD^{-1/2}$
$= (D^{-1/2})^T V^TA^TAVD^{-1/2}$ ∴ $A^T = VD$
$= D^{-1/2} \cancel{V^TV} D \cancel{V^TV} D^{-1/2}$ ∴ $A = DV^T$
$= I$

Thus, setting $U = AVD^{-1/2}$, we find U has orthonormal cols.
Massage this: $U = AVD^{-1/2}$
$UD^{1/2} = AV$
$UD^{1/2}V^T = A$
$U\Sigma V^T = A$

$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}$