Examples of time-reversible MC's:

Ex) 2-state

\[ P = \begin{bmatrix} \frac{1}{q+p} & \frac{p}{q+p} \\ \frac{1}{q+p} & \frac{q}{q+p} \end{bmatrix} \]

Reversible?

\[ P_0 \cdot P_{ij} = P_{ji} \cdot P_{0j} \]

LHS = \frac{q}{q+p} \cdot \frac{1}{q+p} \quad \text{RHS} = \frac{p}{q+p} \cdot \frac{q}{q+p} = \text{LHS} \quad \checkmark

Ex)

\[ P = \begin{bmatrix} 0 & 1-p & p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{bmatrix} \]

\[ \Pi = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad \text{(by symmetry)} \]

\[ \text{(unique since it is finite-state)} \quad \text{irreducible.} \]

Time reversible?

\[ \Pi_i P_{ij} = \Pi_j P_{ji} \quad \text{only if } p = \frac{1}{2} \]

Ex) Ehrenfest chain

M molecules divided between 2 urns.
At each step, select a molecule at random & move it to other urn.
Let \( X_n \) = # of molecules in urn 1.
State space: \( E_0, \ldots, M \) 

Transition probs:

\[ P_{i,i+1} = \frac{M - i}{M} \quad \text{gain molecule} \]

\[ P_{i,i-1} = \frac{i}{M} \quad \text{lose molecule} \]

Claim: Any irreducible finite state 1-d RW, w/ arbitrary transition probs, is time reversible.

\[ P_{i,j} = P_{j,i} \quad (i = j \pm 1) \]

Asymptotically, no matter how long MC has been running, the number of transitions from \( i \) to \( j \) is within 1 of

\[ 11 \quad 11 \quad 11 \quad j \text{ to } i \]

\[ \Rightarrow \text{proportions are asymptotically equal} \]

Goal: Find \( \pi \), i.e., solve

\[ \pi_i P_{ij} = \pi_j P_{ji} \quad \sum_{i} \pi_i = 1 \]

Method 1: Guess and verify.

\[ \Rightarrow B_1 n \left( M \frac{1}{2} \right) \]