- Prove finite state irreducible MC has unique stationary dist.

- Talk about why we care about stationary dist.

- Ask about intuition for time reversal.

**Time Reversal**

**Theorem:** Given an MC \((X_n)_{n \geq 0}^N\) w/ stationary dist \(\pi\) & \(P(X_0 = i) = \pi_i\), let \(Y_n = X_{N-n}\).

Then \((Y_n)_{n \geq 0}^N\) is a MC w/ stationary dist \(\pi\) and transition probs \(Q_{ij} = \frac{P_{ji} \pi_j}{\pi_i}\).

**Proof:**

**Claim 1:** \(Y_n\) is MC.

We need to show Markov property:

\[ P(Y_n = i | Y_{n-1} = j, Y_{n-2} = k, \ldots) = P(Y_n = i | Y_{n-1} = j) \]

\[ \text{RHS} = P(X_{N-n} = i | X_{N-n+1} = j) \]

\[ \text{LHS} = P(X_{N-n} = i | X_{N-n+1} = j, X_{N-n+2} = k, \ldots) \]

\[ = \frac{P(X_{N-n+2} = k, \ldots | X_{N-n} = i, X_{N-n+1} = j) \cdot P(X_{N-n+1} = i | X_{N-n+2} = k, \ldots)}{P(X_{N-n+1} = k, \ldots | X_{N-n} = i, X_{N-n+1} = j)} \]

\[ = \text{RHS}, \]

**Claim 2:** Transition probs:

\[ Q_{ij} = P(Y_n = j | Y_{n-1} = i) = P(X_{N-n} = j | X_{N-n+1} = i) \]
\[ P(X_{n+1} = i \mid X_n = j) \cdot P(X_n = j) \]
\[ = P_{j,i} \cdot \frac{P_{i,j}}{\pi_i} \]

Since \( X_n \) starts in stationary dist. If it didn’t we wouldn’t have time homogeneity.

Claim 3: \( \pi \) is stationary for \( \pi \) if
\[
\text{row sum} \rightarrow 1
\]

Indeed \[
(\pi \cdot Q)j = \sum_i P_{j,i} \pi_i = \sum_i \frac{\pi_j}{\pi_i} \pi_i = \pi_j \cdot 1 = \pi_j
\]
\[ \Rightarrow \pi \cdot Q = \pi. \]

Def. A time reversible MC has \( P_{ij} = Q_{ij} \)

Proof Let \( X_n \) be an irreducible MC. If \( \exists \, x = (x_i) \)
\[ \sum_i x_i = 1 \quad \text{and} \quad x_i P_{ij} = x_j P_{ji} \]
then \( x = \pi \) is the stationary dist.

Pf: \[
(xP)j = \sum_i x_i P_{ij} = \sum_i x_j P_{ji} = x_j \quad \Rightarrow \quad xP = x.
\]

Also, note \( \sum_i x_i = 1 \quad \forall \, i. \) (why?)