Isoperimetric $\neq$, Concentration of Lip. functions

Note Title 2015-09-23

Remarks:

1. Any equatorial band of width $\sim \frac{1}{\sqrt{n}}$ has constant measure: $\sigma(E_\epsilon) = .99$ for $\epsilon \leq \frac{c}{\sqrt{n}}$

2. Equivalent form of prop: for a hemisphere, H, its ϵ-neighborhood A_ϵ has large measure:

$$\sigma_n(A_\epsilon) \geq 1 - \exp\left(\frac{n\epsilon^2}{2}\right) \quad (*)$$

In fact, 2 happens for any set with

measure $\geq \frac{1}{2}$!

This follows from:

$\boxed{\text{Thm (Isoperimetric} \neq \text{for } S^{n-1}\text{)}}$

Among all measurable sets $A \subseteq S^{n-1}$ with a given measure, and for given $\epsilon > 0$, spherical caps minimize the measure of $A_\epsilon = \{x \in S^{n-1} : \mathrm{dist}(x, A) \leq \epsilon\}$

$\Downarrow$

Thm (Concentration on S^{n-1}) Every measurable set $A \subseteq S^{n-1}$ with $\sigma_{n-1}(A) \geq \frac{1}{2}$ satisfies

$$\sigma_{n-1}(A_\epsilon) \geq 1 - \exp\left(\frac{n\epsilon^2}{2}\right), \qquad \epsilon > 0$$

Proof: $\sigma_{n-1}(A_\epsilon) \geq \sigma_{n-1}(\overset{\text{hemisphere}}{H_\epsilon})$ (by isoperimetric $\neq$)

$$\geq 1 - \exp\left(\frac{n\epsilon^2}{2}\right) \quad (\text{by } (*))$$

QED

<u>Remark</u> on isoperimetric $\neq$: Let $A \subseteq S^{n-1}$ be measurable and Cap be a spherical cap satisfying $\sigma_{n-1}(A) = \sigma_{n-1}(\text{Cap})$. The area of the <u>boundary</u> of A is

$$\sigma_{n-2}(A) := \lim_{\epsilon \to 0} \frac{\sigma_{n-1}(A_\epsilon) - \sigma_{n-1}(A)}{\epsilon},$$

$$\geq \lim_{\epsilon \to 0} \frac{\sigma_{n-1}(\text{Cap}_\epsilon) - \sigma_{n-1}(\text{Cap})}{\epsilon} \quad (\text{By isop. } \neq)$$

$$= \sigma_{n-1}(\mathrm{Cap})$$

Cor (Classical isoperimetric $\neq$ for S^{n-1})

Among all sets in S^{n-1} of the same measure, spherical caps minimize the area of the boundary.

- Recall a similar phenomenom in $\mathbb{R}^n$: Euclidean balls minize ratio of surface area to volume.

Literature:

- R. Vershynin: Lectures in geometric funct. anal.
- M. Ledoux: Concentration of measure phenomenon
- K. Ball: An elementary intro. to modern conv. geom.

2.2 Concentration for Lipschitz functions

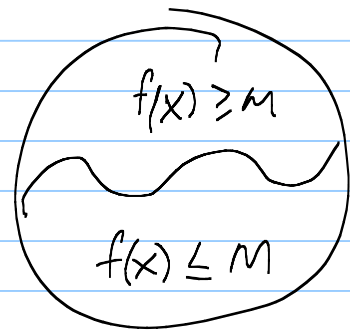
Consider a Lipschitz function $f : S^{n-1} \to \mathbb{R}$ w/ Lip. const. L, so $|f(x) - f(y)| \leq L\|x-y\|_2$,

w/ Euclidean metric

- Goal: Show that f is concentrated around one value M.

- $M = \text{median}(f)$, defined by

$$\sigma_{n-1}(\{x : f(x) \leq M\}) \geq \tfrac{1}{2}, \quad \sigma_{n-1}(\{x : f(x) \geq M\})$$



- Let $A = \{x : f(x) \leq M\}$ so $\sigma_{n-1}(A) \geq \frac{1}{2}$.

  By concentration of measure thm,

$$(*)\ \sigma_{n-1}(A_\epsilon) \geq 1 - \exp\left(-\frac{n\epsilon^2}{2}\right), \quad \epsilon > 0$$

- Note that for each $x \in A_\epsilon$, $\exists\, y \in A$ s.t. $\|x-y\|_2 \le \epsilon$. Thus

$$f(x) \le f(y) + L\cdot\epsilon \le M + L\cdot\epsilon \quad (\text{since } y \in A)$$

Thus, $A_\epsilon \subseteq \{x \in S^{n-1} : f(x) \le M + L\cdot\epsilon\}$

- By $(*)$, $\sigma_{n-1}(x \in S^{n-1} : f(x) \le M + L\cdot\epsilon) \ge 1 - \exp\left(-\frac{n\epsilon^2}{2}\right)$

- Similar argument for $B = \{x \in S^{n-1} : f(x) \ge M - L\cdot\epsilon\}$

$$\Rightarrow \sigma_{n-1}(\{x \in S^{n-1} : |f(x) - M| \ge L\epsilon\}) \le 2\exp\left(-\frac{n\epsilon^2}{2}\right)$$

We have proved:

Thm Let $f: S^{n-1} \to \mathbb{R}$ have Lip. const. L, $M = \text{median}(f)$.

$$\boxed{\sigma_{n-1}\left(\{x \in S^{n-1} : |f(x) - M| \geq \epsilon\}\right) \leq 2\exp\left(\frac{-n\epsilon^2}{2L^2}\right)}$$

(we replaced $L\epsilon$ w/ ϵ)

"A Lipschitz function is almost constant on almost the entire sphere" – Milman?

Exercize: Consider $f(x) = \|x\|_1$.

2.3 Concentration in Gauss space

$(S^{n-1}, \|\cdot\|_2, \sigma_{n-1})$ can be replaced with

$(\mathbb{R}^n, \|\cdot\|_2, \gamma_n)$

$\uparrow$

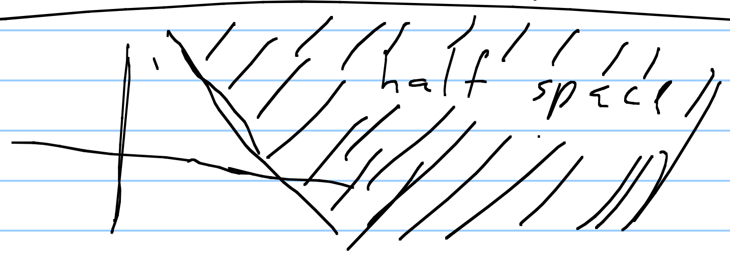
Gauss measure w/ density $f(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{\|x\|_2^2}{2}}$.

so $\gamma_n(A) = \int_A f(x)\,dx$

Argument has same major steps:

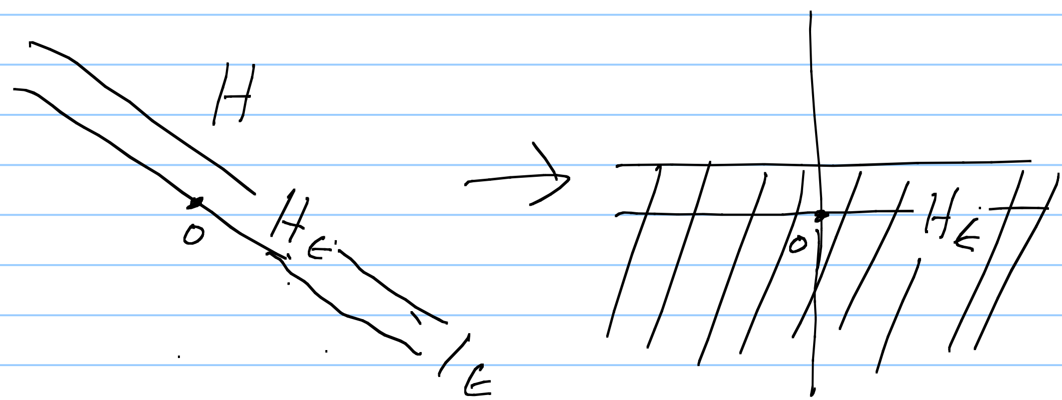
① **Thm** (Isoperimetric ≠ in Gauss space)

Among all measurable sets $A \subseteq \mathbb{R}^n$ w/ a given measure half-spaces minimize $\gamma_n(A_\epsilon)$



② <u>Consider $\gamma_n(H) = \frac{1}{2}$</u>

If H is a half-space with $\gamma_n(H) = \frac{1}{2}$ then by rotation invariance



$$\gamma_n(H_\epsilon) = \gamma_n\{x \in \mathbb{R}^n : x_1 \le \epsilon\}$$
$$= P(N(0,1) \le \epsilon)$$
$$\ge 1 - \exp\left(-\frac{\epsilon^2}{2}\right) \quad \text{(Gaussian tail bound)}$$

Combine ① & ② $\Rightarrow$

Thm: (<u>Concentration in Gauss space</u>)

Every measurable set $A \subseteq \mathbb{R}^n$ satisfies
$$\gamma_n(A_t) \ge 1 - \exp\left(\frac{-t^2}{2}\right) \qquad t > 0$$

Thm (Functional form) Let $f: \mathbb{R}^n \to \mathbb{R}$ have Lipschitz const. L, $\mathrm{med}(f) = M$. Then

$$\gamma_n\left(x : |f(x) - M| > t\right) \le 2\exp\left(-\frac{t^2}{2L^2}\right)$$

Exercize: prove these 2 theorems.

Probabilistic interpretation: Let $g = (g_1, \ldots g_n) \sim N(0, I_n)$ so $g_i \overset{iid}{\sim} N(0,1)$. Then

$$P\left(|f(g) - M| > t\right) \le 2\exp\left(-\frac{t^2}{2L}\right).$$

In other words, $f(g) - M$ is sub-Gaussian w/ $\|f(g) - M\|_{\psi_2} \le CL$.

Centering: $\|f(g) - \mathbb{E} f(g)\|_{\psi_2} \le CL$.

<u>Corollary</u>: A Lipschitz function of a Gaussian vector is sub-Gaussian. In particular, let $f: \mathbb{R}^n \to \mathbb{R}$ have Lip. const. L, $g \sim N(0, I_n)$. Then

$$\|f(g) - \mathbb{E} f(g)\|_{\psi_2} \le CL$$

$\boxed{\text{HW}}$: Give a similar corollary for a Lipschitz function of a vector chosen uniformly from S^{n-1}.