Sub-Gaussian r.v.'s continued

Note Title 2015-09-16

Recall: sub-Gaussian properties:

① Tails: $P(|X|>t) \le \exp\left(-\frac{t^2}{K_1^2}\right)$ $t \ge 0$

② Moments: $(\mathbb{E}|X|^p)^{1/p} \le K_2\sqrt{p}$ $p \ge 1$

③ Tails': $\mathbb{E}\exp\left(\frac{X^2}{K_3^2}\right) \le e.$

④ MGF: If $\mathbb{E}X=0$, then

$$\mathbb{E}\exp(tX) \le e^{t^2K_4^2} \quad t \in \mathbb{R}$$

How to compare "size" of sub-Gauss r.v.'s?

$$\boxed{\begin{array}{l}\text{\underline{Def} The \underline{sub-Gaussian norm} of a r.v. } X \text{ is the smallest } K_2 \\ \text{in property ②:} \\ \|X\|_{\psi_2} := \sup p^{-\frac{1}{2}} (\mathbb{E}|X|^p)^{\frac{1}{p}}\end{array}}$$

Note: • X is sub-Gaussian $\iff \|X\|_{\psi_2} < \infty$

• $\|X\|_{\psi_2}$ plays similar role as standard deviation for mean-zero r.v.'s.

Exercise: Show that $\|\cdot\|_{\psi_2}$ is indeed a norm, defining the space of sub-Gaussian r.v.'s.

Corollary Let X be a sub-Gaussian r.v.. Then

① $P(|X|>t) \le \exp\left(1-\frac{ct^2}{\|X\|_{\psi_2}^2}\right)$, $\quad t\ge 0$

② $(\mathbb{E}|X|^p)^{\frac{1}{p}} \le \|X\|_{\psi_2}$ $\quad p\ge 1$

③ $\mathbb{E}\exp\left(c\frac{X^2}{\|X\|_{\psi_2}^2}\right)\le e$

④ If $\mathbb{E}X=0$, then

$\mathbb{E}\exp(tX)\le \exp(Ct^2\|X\|_{\psi_2}^2)$ $\quad t\in\mathbb{R}$

C, c: abs. const's.

$\|X\|_{\psi_2}$ is the smallest number possible in these $\neq$'s (up to abs. const's).

Examples of sub-Gaussian r.v.'s:

- Gaussian: $g\sim N(0,1)$, $\|g\|_{\psi_2}=C$

  More generally, $g\sim N(0,\sigma^2)$, $\|g\|_{\psi_2}=C\sigma$

- Rademacher:

  $|X|=1 \Longrightarrow \mathbb{E}|X|^p=1 \Rightarrow \|X\|_{\psi_2}=1$

- More generally, for any bounded r.v.,

$\|X\|_{\psi_2} \leq \|X\|_\infty$ (Recall: $|X| \leq \|X\|_\infty$ a.s.)

Not Sub-Gaussian:

- Exponential r.v.'s: $X \sim Exp(\lambda)$

$\Rightarrow P(|X|>t) = \underbrace{\exp(-\lambda t)}$

exponential tail does not decay fast enough!

Others: Geometric, Cauchy, Poisson...

"Size" of sum of r.v.s.

Note: If $g_i \sim N(0, \sigma_i^2)$ then

$$\sum_{i=1}^n g_i \sim N\left(0, \sum_{i=1}^n \sigma_i^2\right)$$

$\underline{\text{Lemma}}$ (Sum of sub-Gaussian r.v.'s)

Let $X_1, \ldots X_n$ be ind. sub-Gaussian r.v.'s with $\mathbb{E} X_i = 0$. Then $\sum_{i=1}^n X_i$ is sub-Gaussian and

$$\left\| \sum_{i=1}^n X_i \right\|_{\psi_2}^2 \leq c \sum_{i=1}^n \|X_i\|_{\psi_2}^2 \qquad (*)$$

Proof: Use MGF:

$$\mathbb{E}\exp\left(t\sum_{i=1}^{n}X_i\right) = \prod \mathbb{E}\exp(tX_i)$$

$$\leq \prod \exp\left(C t^2 \|X_i\|_{\psi_2}^2\right)$$

$$= \exp\Big(t^2 \, c \underbrace{\sum_{i=1}^{n}\|X_i\|_{\psi_2}^2}_{K_4^2 \text{ from property } 4}\Big)$$

Note: K_4 is equivalent to $\|\cdot\|_{\psi_2}$,

i.e. $\left\|\sum_{i=1}^{n} X_i\right\|_{\psi_2} \leq C K_4 = C\sqrt{\sum_{i=1}^{n}\|X_i\|_{\psi_2}^2}$

QED

Thm: Hoeffding ≠ for sub-Gauss. r.v.'s

Let X_i be mean-zero, sub-Gauss. r.v.'s. Then for all $\{a_1, \dots a_n\} \in \mathbb{R}^n$,

$$P\left(\left|\sum_{i=1}^{n} a_i X_i\right| \geq t\right) \leq e \cdot \exp\left(\frac{-t^2}{c\sum_{i=1}^{n} a_i^2 \|X\|_{\psi_2}^2}\right)$$

Proof: Exercise. (~2 lines)

Q: Agrees (roughly) with CLT?

Yes, if $\sqrt{Var(X_i)} \approx \|X_i\|_{\psi_2}$

Q: When is this <u>not</u> true?

Comparison of $\sqrt{Var\left(\sum_i X_i\right)}$, $\|\sum_i X_i\|_{\psi_2}$

given by:

<u>Thm</u> <u>Khintchine ≠</u>. Let $X_1, \dots X_n$ be ind., sub. Gauss. r.v.'s, w/ $\mathbb{E} X_i = 0$, $Var(X_i) = 1$. Then $\forall \{a_1, \dots a_n\} \in \mathbb{R}^n$, $p \geq 2$,

$$\sqrt{\sum_{i=1}^n a_i^2} \leq \left(\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|^p\right)^{1/p} \leq c \sqrt{\sum_{i=1}^n a_i^2 \|X_i\|_{\psi_2}^2} \sqrt{p}$$

Proof: Lower bound:

$$\left(\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|^p\right)^{\frac{1}{p}} = \left(\mathbb{E}\left(\left|\sum_{i=1}^n a_i X_i\right|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

$$\leq \sqrt{\mathbb{E}\left|\sum_{i=1}^n a_i X_i\right|^2} \quad \text{(by Jensen's ≠)}$$

$$= \sqrt{Var\left(\sum_i a_i X_i\right)}$$

$$= \sqrt{\sum_i a_i^2 \, Var(X_i)}$$

$$= \sqrt{\sum_{i=1}^n a_i^2} \quad \checkmark$$

Upper bound: <u>Exercize.</u>

Q: What if $\mathbb{E}X_i \neq 0$?

Remark: (Centering) We typically assume $\mathbb{E}X = 0$ (X_i is centered).

If not, we may replace X w/

$$Y := X - \mathbb{E}X.$$

Note: $\mathbb{E}Y = 0$

Further, if X is sub-Gaussian, then so is Y and

$$\|Y\|_{\psi_2} \leq 2\|X\|_{\psi_2}$$

Proof: $\|Y\|_{\psi_2} := \|X - \mathbb{E}X\|_{\psi_2}$

$$\leq \|X\|_{\psi_2} + \|\mathbb{E}X\|_{\psi_2}$$

$\mathbb{E}X$ is a const, so

$$\|\mathbb{E}X\|_{\psi_2} = |\mathbb{E}X| \underset{\substack{\uparrow \\ \text{Jensen} \neq}}{\leq} \mathbb{E}|X| \leq \|X\|_{\psi_2}$$

QED