Recall: Let $X$ be CTMC in stat. dist. $Z$ be reverse time CTMC.

Then $Z$ leaves state $i$ at rate $V_i$ (same as $X$)

- $Z$ has trans. matrix $Q_{ij} = \frac{P_{ij}b_i}{P_{ij}}$

It is time reversible if

$P_{ij} = Q_{ij}$

$\Rightarrow \pi_iP_{ij} = \pi_jP_{ji}$

$\Rightarrow P_iV_iP_{ij} = P_jV_jP_{ji}$ since $P_i = \frac{1}{2} \pi_i$

Def. (Detailed balance eq.)

$P_i q_{ij} = P_j q_{ji}$

rate of jumps rate of jumps
from $i \rightarrow j$ from $j \rightarrow i$

As in the case of discrete time MC's, this yields a way to find stat. dist.

Prop. Suppose there is a prob. dist. $w = (w_i)$ satisfying the detailed balance eqns, i.e.

$w_i q_{ij} = w_j q_{ji}, \quad \sum_i w_i = 1, \quad w_i > 0.$

Then the CTMC is reversible and $P_j = w_j$.
An ergodic birth death process is time reversible.

Why? Analog to 1-d random walk.

Q: Consider an M/M/1 queue w/ \( \lambda < \mu \).
   This is a birth death process w/ 
   \[
   \lambda_n \quad M_n = \sum_{n \leq s} M_n \quad n \geq s
   \]

Suppose it is has been running a long time and has achieved equilibrium. Let \( N(t) \) count the number of departures from a certain time \( T \gg 0 \). What is dist of this counting proc?

Soln.: Arrivals occur according to rate-\( d \) Poisson process. M/M/1 queue is reversible, i.e., is the same process going backwards in time (arrivals switch w/ departures). Thus, departures are a rate \( d \) Poisson proc.

Q: Reversibility for 3-state MC?
\[ W_0 \cdot P \cdot V_0 = W_1 \cdot (-P) \cdot V_1 \]

\[ W_1 \cdot P \cdot V_1 = W_2 \cdot (-P) \cdot V_2 \]

\[ W_2 \cdot P \cdot V_2 = W_0 \cdot (-P) \cdot V_0 \]

\[ S_1 = ? \]

Take product of the 3 equations to give:

\[ (w_1 \cdot w_2 \cdot w_3) (v_1 \cdot v_2 \cdot v_3) \cdot P^3 = \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} (-P)^3 \]

\[ S_1 = \text{only if } P = 1 - P = \frac{1}{2} \]