Embedded MC. Let \( \{ X(t) : t \geq 0 \} \) be CTMC w/ transition probs \( \{ P_{ij} \} \) rates of leaving states \( \{ V_i \} \).

Let \( Y_n \) be the \( n \)th state that \( X(t) \) jumps to, i.e. \( Y_n = X(S_n) \) where \( S_n \) is time of \( n \)th jump. This is the embedded MC. It has transition matrix \( P = (P_{ij}) \).

Q: Suppose \( \{ Y_n : n \geq 0 \} \) has stationary dist \( (\pi_0, \pi_1, \ldots) \)

Suppose \( \{ X(t) : t \geq 0 \} \) has limiting probs \( (P_0, P_1, \ldots) \)

Recall: \( P_i = \lim_{t \to \infty} P_{ij}(t) \).

What is relationship between \( \pi_i \) & \( P_i \)?

Observe: \( \pi_i \) = proportion of visits to \( i \) (for \( X \& Y \))

\[ \frac{1}{\pi_i} = \text{avg length of time in state } i \text{ per visit (for } X) \]

Guess \( P_i \) is proportional to \( \frac{\pi_i}{\sqrt{V_i}} \), i.e.

\[ P_i = \frac{1}{Z} \frac{\pi_i}{\sqrt{V_i}} \text{ with } Z = \sum_i \frac{\pi_i}{\sqrt{V_i}} \text{ (X)} \]

This can be verified by checking eqn for \( P_i \):

\[ \sum_{j} V_{ij} P_{ij} = \sum_{k \neq i} \frac{\pi_i}{\sqrt{V_i}} P_{ik} \cdot q_{ij} \text{ rate leaving } i \]
Plug (×) into LHS giving $\frac{1}{2} \cdot \frac{\pi_i}{v_j} = \frac{1}{2} \pi_i$

Plug (×) into RHS giving $\frac{1}{2} \cdot \frac{\pi_i \times P_j}{v_k} = \frac{1}{2} \pi_i P_j = \frac{1}{2} \pi_i$ (since $\pi P = \pi$)

**Conclusion:** If $(\pi_i)$ is stat. dist. for $Y$, then $(P_i = \frac{1}{2} \cdot \frac{\pi_i}{v_i})$ are limiting probs for $X$ & vice versa.